Anna University Syllabus

ELECTROMAGNETIC THEORY

UNIT I: ELECTROSTATICS – I 9

Sources and effects of electromagnetic fields – Coordinate Systems – Vector fields – Gradient, Divergence, Curl – theorems and applications - Coulomb's Law – Electric field intensity – Field due to discrete and continuous charges – Gauss's law and applications.

UNIT II: ELECTROSTATICS – II 9

Electric potential – Electric field and equipotential plots, Uniform and Non-Uniform field, Utilization factor – Electric field in free space, conductors, dielectrics - Dielectric polarization – Dielectric strength - Electric field in multiple dielectrics – Boundary conditions, Poisson's and Laplace's equations, Capacitance, Energy density, Applications.

UNIT III: MAGNETOSTATICS 9

Lorentz force, magnetic field intensity (H) – Biot–Savart's Law - Ampere's Circuit Law – H due to straight conductors, circular loop, infinite sheet of current, Magnetic flux density (B) – B in free space, conductor, magnetic materials – Magnetization, Magnetic field in multiple media – Boundary conditions, scalar and vector potential, Poisson's Equation, Magnetic force, Torque, Inductance, Energy density, Applications.

UNIT IV: ELECTRODYNAMIC FIELDS 9

Magnetic Circuits - Faraday's law – Transformer and motional EMF – Displacement current - Maxwell's equations (differential and integral form) – Relation between field theory and circuit theory – Applications.

UNIT V: ELECTROMAGNETIC WAVES 9

Electromagnetic wave generation and equations – Wave parameters; velocity, intrinsic impedance, propagation constant – Waves in free space, lossy and lossless dielectrics, conductors - skin depth - Poynting vector – Plane wave reflection and refraction – Standing Wave – Applications.

TOTAL (L:45 + T:15): 60 PERIODS

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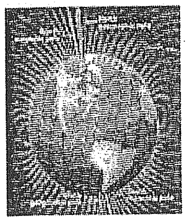
ANNA UNIVERSITY QUESTION PAPERS



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ELECTROSTATICS-I

1.1. INTRODUCTION TO ELECTROMAGNETIC FIELDS

Electromagnetic fields are present everywhere in our environment but are invisible to the human eye. The electromagnetic spectrum encompasses both natural and human made sources of electromagnetic fields. Electric fields are produced by the electric charges in the atmosphere associated with thunderstorm. Magnetic fields are produced by the earth, which causes the compass needle to orient in a North-South direction and is used by binds and fish for navigation. Besides natural sources, the electromagnetic fields are produced by human mode sources such X-ray, Radio, TV, Mobile phone, *etc.*

An electromagnetic field is a physical field produced by electrically charged particles. This field is the combination of an electric field and magnetic field. The electric field is produced by stationary charges (charges at rest), and the magnetic field by moving charges; these two are often described as the sources of field. Electric fields are created by differences in voltage; the higher in voltage difference, the stronger the magnetic field. Magnetic fields are created when the electric current flows: the greater the current, the stronger the magnetic field. An electric field may exist even when there is no current flow.

The sources of electromagnetic fields are electric charges, and the strength of a field at any point depends upon the magnitude, position, velocity and acceleration of the charges involved. An electrostatic field in which the sources are stationary, so that only magnitude and position of the charges need be considered.

1.2. VECTOR ANALYSIS

1.2.1. Introduction

In most of the general electromagnetic field problems, there are three space variables involved, and the solutions tend to become correspondingly complex. The additional complexity that results from having to deal with vector quantities in three directions can largely be overcome by use of vector analysis. For this reason the first topic to be treated will be vector analysis.

Vector analysis is a mathematical shorthand. Any physical quantity can be represented either as a scalar or a vector. ***A quantity that is characterised only by magnitude is called a scalar.*** Examples of ***scalar quantities*** are ***mass, time, temperature*** and ***electric potential.***

A quantity that is characterised both by magnitude and direction is called a vector. Force, velocity, electric field intensity and ***electric flux density*** are the examples of ***vector quantities.*** A vector can be represented geometrically by an arrow whose direction is appropriately chosen. If at each point of a region, there is corresponding value of some physical function, the region is called a field. Fields may be represented as scalar or vector.

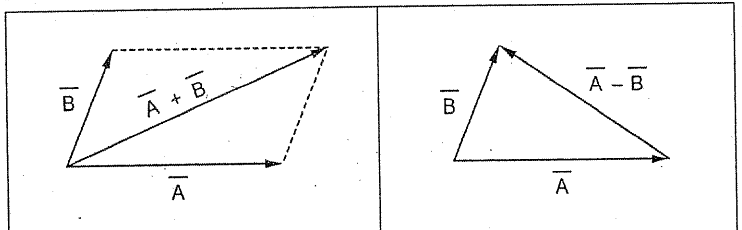
A field is a spatial distribution of a quantity, which may or may not be a function of time. In other words, a field is associated with a region in space and that field exists in the region if there is a physical phenomenon associated with points in that region. Time varying electric and magnetic fields are coupled, resulting in an electromagnetic field.

If a scalar quantity is specified in a region, then the region is said to be a scalar field. Atmospheric temperature and ***pressure*** in a region are the examples of ***scalar field.***

If a vector quantity is specified in a region, then the region is said to be a vector field. The examples of ***vector field*** are ***gravitational force*** and ***wind velocity*** in atmosphere.

1.2.2. Sum and Difference of Two Vectors

The sum of two vectors is the resultant of two vectors as shown in the Fig.1.1(*a*). The difference of two vectors is illustrated in Fig.1.1(*b*).



(a) Summation of two vectors ***(a) Difference of two vectors***

Fig. 1.1. Sum and difference of two vectors

Let vectors A and B be represented in three dimensional axes.

$$\overline{A} = A_x\,\overline{a}_x + A_y\,\overline{a}_y + A_z\,\overline{a}_z$$

$$\overline{B} = B_x\,\overline{a}_x + B_y\,\overline{a}_y + B_z\,\overline{a}_z$$

Sum $$\overline{A}+\overline{B} = (A_x+B_x)\,\overline{a}_x + (A_y+B_y)\,\overline{a}_y + (A_z+B_z)\,\overline{a}_z$$

Difference $$\overline{A}-\overline{B} = (A_x-B_x)\,\overline{a}_x + (A_y-B_y)\,\overline{a}_y + (A_z-B_z)\,\overline{a}_z$$

1.2.3. Multiplication of Scalar and a Vector

When a vector is multiplied by a scalar, a new vector is produced whose direction is the same as the original vector and whose magnitude is the product of the magnitudes of the vector and scalar.

$$\overline{A} = a\overline{B}$$

$$\xrightarrow{\overline{B}} \qquad \xrightarrow{a\overline{B}}$$

where A is new vector
B is given vector
a is given scalar

Scalar Multiplication

Scalar multiplication *of two vectors is scalar quantity whose magnitude is the product of the magnitudes of the vectors multiplied by the cosine of the angle between them. It is referred as* ***dot product.***

Let $\overline{A}$ and $\overline{B}$ be two vectors.

The vector product (dot product) is given by

$$\overline{A} \cdot \overline{B} = AB \cos\theta$$

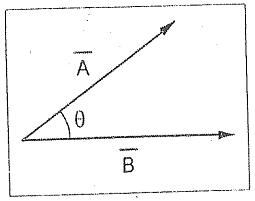


Fig. 1.2.

It obeys commutative law.

$$\overline{A} \cdot \overline{B} = \overline{B} \cdot \overline{A}$$

$$\text{Let,} \quad \overline{A} = A_x \overline{a}_x + A_y \overline{a}_y + A_z \overline{a}_z$$

$$\overline{B} = B_x \overline{a}_x + B_y \overline{a}_y + B_z \overline{a}_z$$

$$\overline{A} \cdot \overline{B} = (A_x \overline{a}_x + A_y \overline{a}_y + A_z \overline{a}_z) \cdot (B_x \overline{a}_x + B_y \overline{a}_y + B_z \overline{a}_z)$$

$$= A_x B_x + A_y B_y + A_z B_z$$

$$\because \quad \overline{a}_x \cdot \overline{a}_x = \overline{a}_y \cdot \overline{a}_y = \overline{a}_z \cdot \overline{a}_z = 1$$

$$\overline{a}_x \cdot \overline{a}_y = \overline{a}_y \cdot \overline{a}_z = \overline{a}_z \cdot \overline{a}_x = 0$$

Vector Multiplication

The vector product of two vectors is defined as a vector whose magnitude is the product of the magnitude of the two vectors and the sine of the angle between them. This multiplication is called, ***"cross product"***.

Let A and B be two vectors.

$$A = A_x \overline{a}_x + A_y \overline{a}_y + A_z \overline{a}_z$$

$$B = B_x \overline{a}_x + B_y \overline{a}_y + B_z \overline{a}_z$$

where $\overline{a}_x$, $\overline{a}_y$ and $\overline{a}_z$ are unit vectors in the direction of x, y and z respectively.

$$|A \times B| = AB \sin\theta$$

$$A \times B = \begin{vmatrix} \overline{a}_x & \overline{a}_y & \overline{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The vector B × A has the same magnitude but opposite direction.

$$B \times A = -A \times B$$

1.2.4. Differentiation

The differential vector operator ∇ is called del or nabla defined as

$$\nabla = \frac{\partial}{\partial x}\overline{a}_x + \frac{\partial}{\partial y}\overline{a}_y + \frac{\partial}{\partial z}\overline{a}_z$$

There are three possible operations with ∇.

1. Gradient: *The gradient of any scalar function is the maximum space rate of change of that function. If the scalar V represents electric potential, ∇V represents potential gradient.*

$$\nabla V = \frac{\partial V}{\partial x}\overline{a}_x + \frac{\partial V}{\partial y}\overline{a}_y + \frac{\partial v}{\partial z}\overline{a}_z$$

This operation is called the gradient.

$$\nabla V = \text{grad } V$$

Gradient of a scalar is a vector.

2. Divergence: *The divergence of a vector 'A' at any point is defined as the limit of its surface integrated per unit volume as the volume enclosed by the surface shrinks to zero.*

$$\nabla \cdot A = \underset{v \to 0}{\text{Lt}} \frac{1}{v} \oint\!\!\!\oint_S A \cdot \overline{n} \cdot ds$$

It can be expressed as

$$\nabla \cdot A = \left(\overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y} + \overline{a}_z \frac{\partial}{\partial z}\right)\left(\overline{a}_x A_x + \overline{a}_y A_y + \overline{a}_z A_z\right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This operation is called divergence.

$$\nabla \cdot A = \text{div } A$$

Divergence of a vector is a scalar.

Divergence indicates the spreading or diverging of quantity from a point.

Examples: (i) Leaking air from a ballon yields positive divergence.

(ii) Rushing of air into drum under the carriage of a train yields negative divergence.

3. *Curl:* The curl of a vector 'A' at any point is defined as the limit of its surface integral of its cross product with normal over a closed surface per unit volume as the volume shrinks to zero.

$$|\text{curl A}| = \underset{v \to 0}{\text{Lt}} \frac{1}{v} \oint\!\!\!\oint_S \bar{n} \times \text{A} \cdot ds$$

It can be expressed as

$$\nabla \times \text{A} = \left(\frac{\partial \text{A}_z}{\partial y} - \frac{\partial \text{A}_y}{\partial z}\right)\bar{a}_x + \left(\frac{\partial \text{A}_x}{\partial z} - \frac{\partial \text{A}_z}{\partial x}\right)\bar{a}_y + \left(\frac{\partial \text{A}_y}{\partial x} - \frac{\partial \text{A}_x}{\partial y}\right)\bar{a}_z$$

$$\nabla \times \text{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{A}_x & \text{A}_y & \text{A}_z \end{vmatrix}$$

This operation is called curl.

$$\nabla \times \text{A} = \text{Curl A}$$

Curl indicates a measure of the tendency of a vector to rotate or twist.

Example: When a leaf floats in sea water and its rotation is about Z-axis, the curl of velocity is in z direction.

1.2.5. Solenoidal and Irrotational Vectors

A vector $\bar{A}$ is said to be solenoidal if its divergence is zero.

i.e., if $\nabla \cdot \bar{\text{A}} = 0$, then $\bar{\text{A}}$ is said to be solenoidal.

A vector $\bar{A}$ is said to be irrotational if its curl is zero.

i.e., if $\nabla \times \bar{\text{A}} = 0$, then $\bar{\text{A}}$ is said to be irrotational.

1.3. CO-ORDINATE SYSTEMS

Co-ordinate system is a system which is used to represent a point in space. There are three simple co-ordinate systems to describe a vector accurately such as specific length, directions, angles, projections or components. The three most common co-ordinate systems are:

(*i*) Rectangular or Cartesian co-ordinate (x, y, z) system.

(*ii*) Cylindrical co-ordinate (ρ, ϕ, z) system.

(*iii*) Spherical co-ordinate (r, θ, ϕ) system.

Co-ordinate system	Co-ordinates
Cartesian	(x, y, z)
Cylindrical	(ρ, ϕ, z)
Spherical	(r, θ, ϕ)

1.3.1. Cartesian Co-ordinate System

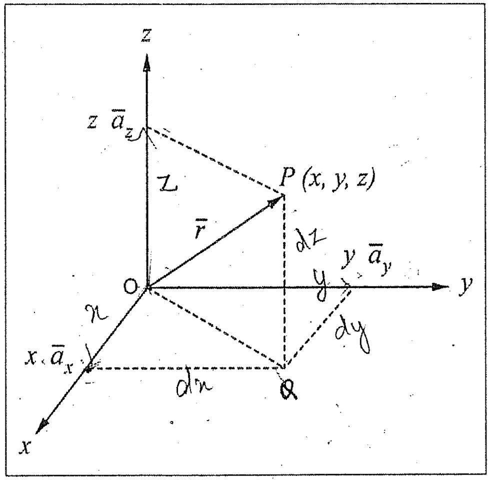


Fig. 1.3. Cartesian co-ordinate system

In cartesian co-ordinate system, three co-ordinates axes x, y, and z are mutually right angles to each other. Consider a point P(x, y, z) in space at a distance r from the origin. The vector r can be represented as

$$\bar{r} = x\,\bar{a}_x + y\,\bar{a}_y + z\,\bar{a}_z$$

where $\bar{a}_x, \bar{a}_y$, and $\bar{a}_z$ are unit vectors.

x, y and z are the components vectors. Components vectors have a magnitude and direction. Unit vectors have unit magnitude and directed along the co-ordinate axis.

A unit vector in a given direction is a vector in that direction divided by its magnitude. It is given by

$$\bar{a}_r = \frac{\bar{r}}{|r|} = \frac{x\,\bar{a}_x + y\,\bar{a}_y + z\,\bar{a}_z}{\sqrt{x^2+y^2+z^2}}$$

Consider the points $P(x, y, z)$ and $Q(x + dx, y + dy, z + dz)$ in rectangular co-ordinate system. Fig.1.3 shows the six planes define a rectangular parallel piped. The differential length dl from P to Q is the diagonal of the parallel piped is given by

$$dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

The differential area, $ds = dx\,dy = dy\,dz = dz\,dx$

The differential volume, $dv = dx\,dy\,dz$

1.3.2. Cylindrical Co-ordinate System

The cylindrical co-ordinate system is the three dimensional version of the polar co-ordinates of analytic geometry. In this system, consider any point P as the intersection of three mutually perpendicular surfaces. They are a circular cylinder (ρ = constant), a plane (ϕ = constant) and another plane (z = constant). This is illustrated in Fig.1.4. The co-ordinates are ρ, ϕ and z.

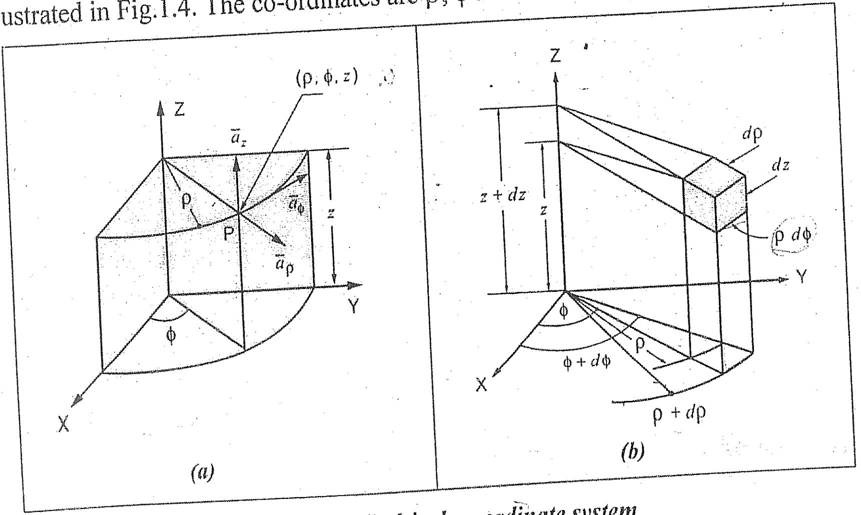


Fig. 1.4. Cylindrical co-ordinate system

A differential volume element in cylindrical co-ordinate may be obtained by increasing ρ, ϕ, and z by the differential increments $d\rho$, $d\phi$ and dz as shown in Fig.1.4. The shape of this small volume is truncated wedge. As the volume element becomes very small, its shape approaches that of a rectangular parallelopiped. It has sides of the length $d\rho$, $\rho\,d\phi$ and dz.

The differential length, $dl = \sqrt{(d\rho)^2 + (\rho\,d\phi)^2 + (dz)^2}$

The differential area, $ds = \rho\,d\rho\,d\phi$

$= d\rho\,dz$

$= \rho\,d\phi\,dz$

The differential volume, $dv = \rho\,d\rho\,d\phi\,dz$

1.3.3. Spherical Co-ordinate System

In this system, consider any point P as the point of intersection of the spherical surface (radius r = constant) conical surface (θ, angle between r and z = constant) and plane surface (ϕ = constant). The co-ordinates of this system are r, θ, and ϕ. Fig.1.5 depicts the co-ordinates of spherical system.

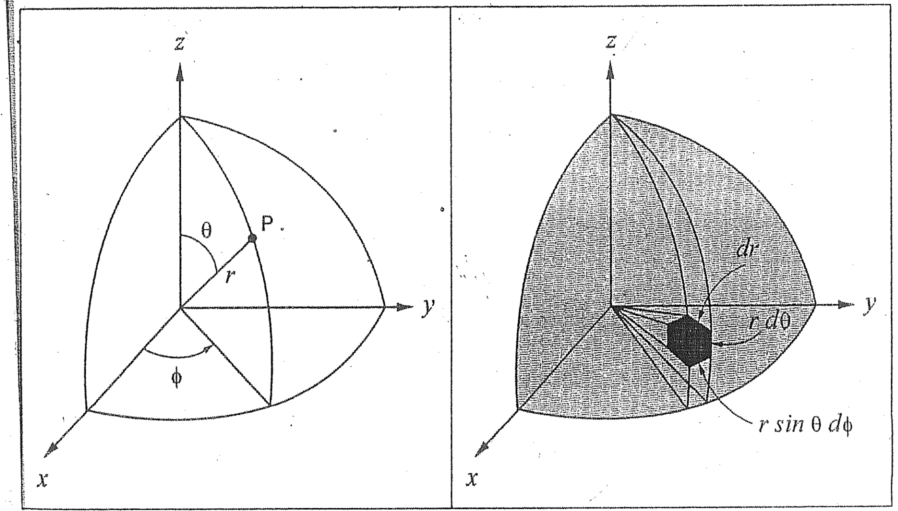


Fig. 1.5. The spherical co-ordinate system

A differential volume element may be obtained in spherical co-ordinate by increasing r, θ, and ϕ by dr, $d\theta$ and $d\phi$ as shown in Fig.1.5. The sides of this volume element are dr, $r\,d\theta$ and $r \sin\theta\, d\phi$.

The differential length, $dl = \sqrt{dr^2 + (r\,d\theta)^2 + (r \sin\theta\, d\phi)^2}$

The differential area,
$$\begin{aligned} ds &= dr \cdot r\,d\theta = r\,dr\,d\theta \\ &= dr \cdot r \sin\theta\, d\phi \\ &= r \sin\theta\, d\phi\, dr \\ &= r\,d\theta \cdot r \sin\theta\, d\phi \\ &= r^2 \sin\theta\, d\theta\, d\phi \end{aligned}$$

The differential volume,
$$\begin{aligned} dv &= dr \cdot r\,d\theta \cdot r \sin\theta\, d\phi \\ dv &= r^2 \sin\theta\, d\theta\, d\phi\, dr \end{aligned}$$

1.3.4. Transformation between Cartesian and Cylindrical Systems

It is necessary to transform a vector from one co-ordinate system to another co-ordinate system.

A vector in cartesian co-ordinate system can be transformed into cylindrical co-ordinate system and vice-versa.

(i) *Conversion of Cartesian to Cylindrical System*

The cartesian co-ordinates (x, y, z) can be converted into cylindrical coordiantes (ρ, ϕ, z).

Given	*Transform*
x	$\rho = \sqrt{x^2 + y^2}$
y	$\phi = \tan^{-1}(y/x)$
z	$z = z$

(ii) *Conversion of Cylindrical to Cartesian system*

The Cylindrical co-ordinates (ρ, ϕ, z) can be converted into Cartesian co-ordinates (x, y, z).

Given	*Transform*
ρ	$x = \rho \cos\phi$
ϕ	$y = \rho \sin\phi$
z	$z = z$

1.3.5. Transformation between Cartesian and Spherical Systems

A vector in cartesian co-ordinate system can be transformed into cylindrical co-ordinate system and vice-versa.

(i) *Conversion of Cartesian to Spherical System*

The Cartesian co-ordinates (x, y, z) can be transformed into spherical co-ordinates (r, θ, ϕ).

Given	*Transform*
x	$r = \sqrt{x^2 + y^2 + z^2}$
y	$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \cos^{-1}(z/r)$
z	$\phi = \tan^{-1}(y/x)$

(ii) *Conversion of Spherical to Cartesian co-ordinate system*

The spherical co-ordinates (r, θ, ϕ) can be transformed into Cartesian co-ordinates (x, y, z).

Given	*Transform*
r	$x = r \sin\theta \cdot \cos\phi$
θ	$y = r \sin\theta \cdot \sin\phi$
ϕ	$z = r \cos\theta$

1.3.6. Summary of the Co-ordinate Systems

The summary of the co-ordinate systems (cartesian, cylindrical and spherical) are shown in Table 1.1.

Table 1.1. Summary of the co-ordinate system

Co-ordinate Variables	**Cartesian co-ordinates**	**Cylindrical co-ordinates**	**Spherical co-ordinates**
Vector representation (A)	$A_x\bar{a}_x + A_y\bar{a}_y + A_z\bar{a}_z$	$A_\rho\bar{a}_\rho + A_\phi\bar{a}_\phi + A_z\bar{a}_z$	$A_r\bar{a}_r + A_\theta\bar{a}_\theta + A_\phi\bar{a}_\phi$
Magnitude | A |	$\sqrt{A_x^2 + A_y^2 + A_z^2}$	$\sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$	$\sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$
Vector properties	$\bar{a}_x\cdot\bar{a}_x = \bar{a}_y\cdot\bar{a}_y = \bar{a}_z\cdot\bar{a}_z = 1$ $\bar{a}_x\cdot\bar{a}_y = \bar{a}_y\cdot\bar{a}_z = \bar{a}_z\cdot\bar{a}_x = 0$ $\bar{a}_x\times\bar{a}_y = \bar{a}_z$ $\bar{a}_y\times\bar{a}_z = \bar{a}_x$ $\bar{a}_z\times\bar{a}_x = \bar{a}_y$	$\bar{a}_\rho\cdot\bar{a}_\rho = \bar{a}_\phi\cdot\bar{a}_\phi = \bar{a}_z\cdot\bar{a}_z = 1$ $\bar{a}_\rho\cdot\bar{a}_\phi = \bar{a}_\phi\cdot\bar{a}_z = \bar{a}_z\cdot\bar{a}_\rho = 0$ $\bar{a}_\rho\times\bar{a}_\phi = \bar{a}_z$ $\bar{a}_\phi\times\bar{a}_z = \bar{a}_\rho$ $\bar{a}_z\times\bar{a}_\rho = \bar{a}_\phi$	$\bar{a}_r\cdot\bar{a}_r = \bar{a}_\theta\cdot\bar{a}_\theta = \bar{a}_\phi\cdot\bar{a}_\phi = 1$ $\bar{a}_r\cdot\bar{a}_\theta = \bar{a}_\theta\cdot\bar{a}_\phi = \bar{a}_\phi\cdot\bar{a}_r = 0$ $\bar{a}_r\times\bar{a}_\theta = \bar{a}_\phi$ $\bar{a}_\theta\times\bar{a}_\phi = \bar{a}_r$ $\bar{a}_\phi\times\bar{a}_r = \bar{a}_\theta$
Dot product (A · B)	$A_xB_x + A_yB_y + A_zB_z$	$A_\rho B_\rho + A_\phi B_\phi + A_zB_z$	$A_rB_r + A_\theta B_\theta + A_\phi B_\phi$
Cross product (A × B)	$\begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \bar{a}_\rho & \bar{a}_\phi & \bar{a}_z \\ A_\rho & A_\phi & A_z \\ B_\rho & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \bar{a}_r & \bar{a}_\theta & \bar{a}_\phi \\ A_r & A_\theta & A_\phi \\ B_r & B_\theta & B_\phi \end{vmatrix}$
Differential length (*dl*)	$\bar{a}_x\,dx + \bar{a}_y\,dy + \bar{a}_z\,dz$	$\bar{a}_\rho\,d\rho + \bar{a}_\phi\,d\phi + \bar{a}_z\,dz$	$\bar{a}_r\,dr + \bar{a}_\theta\,d\theta + \bar{a}_\phi\,d\phi$
Differential surface area (*ds*)	$ds_x = \bar{a}_x\,dy\,dz$ $ds_y = \bar{a}_y\,dx\,dz$ $ds_z = \bar{a}_z\,dx\,dy$	$ds_\rho = \bar{a}_\rho\,\rho\,d\phi\,dz$ $ds_\phi = \bar{a}_\phi\,d\rho\,dz$ $ds_z = \bar{a}_z\,\rho\,d\rho\,d\phi$	$ds_r = \bar{a}_r\,r^2\sin\theta\,d\theta\,d\phi$ $ds_\theta = \bar{a}_\theta\,r\sin\theta\,dr\,d\phi$ $ds_\phi = \bar{a}_\phi\,r\,dr\,d\theta$
Differential volume (*dv*)	$dx\,dy\,dz$	$\rho\,d\rho\,d\phi\,dz$	$r^2\sin\theta\,dr\,d\theta\,d\phi$

1.4. LINE, SURFACE AND VOLUME INTEGRALS

The basic laws of electromagnetic fields in terms of integrals of field quantities over various lines (curves), surfaces and volumes in a region.

Line Integral

Let $f(x)$ be a continuous function of x between the limits $x = 0$ and $x = b$ as shown in Fig.1.6.

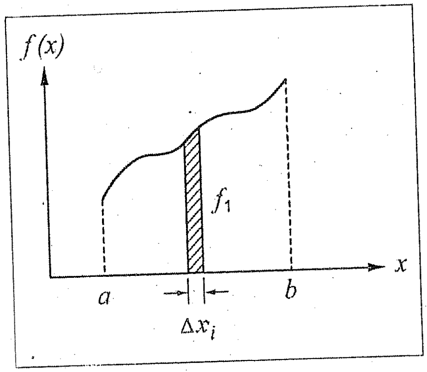


Fig. 1.6.

To define the line integral of $f(x)$, divide the interval from a to b into small segments of length Δx; which approach zero in the limit. The line integral is defined in terms of the limit of the sum as

$$\int_a^b f(x)\,dx = \underset{\substack{n \to \infty \\ \Delta x_i \to 0}}{\text{Lt}} \sum_{i=1}^{n} f_i \Delta x_i$$

where f_i is the value of $f(x)$ for the segment Δx_i. Any field for which the line integral around a closed path is zero is called ***conservative field*** or ***lamellar field.***

$$\oint \text{F} \cdot dx = 0$$

Surface Integral

To define the surface integral of a scalar field or vector field $\overline{\text{F}}$, divide the given surface s into a large number of small surfaces of area Δs_i, which approach zero in

the limit. Each small surface Δs_i has a corresponding vector surface as $\Delta \overline{s_i}$ as shown in Fig.1.7.

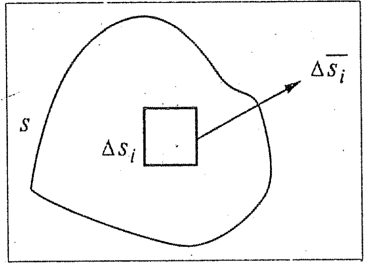


Fig. 1.7.

To define the surface integral of $\overline{\text{F}}$, form the dot product of vector field F with each surface element $\Delta \overline{s}$ and sum it for all n small surfaces of s in the limit $\Delta \overline{s} \to 0$ as $n \to \infty$. The surface integral is defined in terms of the limit of the sum as

$$\int_s \overline{\text{F}} \cdot ds = \underset{\substack{n \to \infty \\ \Delta \overline{s}_i \to 0}}{\text{Lt}} \sum_{i=1}^{n} \overline{\text{F}}_i \, \Delta \overline{s_i}$$

where $\overline{\text{F}}_i$ is the vector field for the small surface $\Delta \overline{s_i}$.

Volume Integral

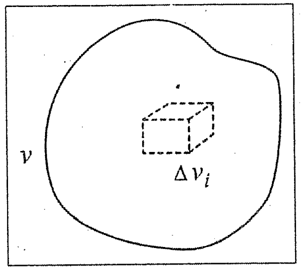


Fig. 1.8.

To define a volume integral, divide a given volume into n small volume elements Δv as shown in Fig.1.8.

To define the volume integral $\overline{\text{F}}_s$, multiply each volume element by the vector field F and sum their products for all n small volumes in the limit $\Delta v \to 0$ as $n \to \infty$. The volume integral is defined in terms of the limit of the sum as

$$\int_v \overline{\text{F}} \cdot dv = \underset{\substack{n \to \infty \\ \Delta v_i \to 0}}{\text{Lt}} \sum_{i=1}^{n} \overline{\text{F}}_i \, \Delta v_i$$

1.5. DIVERGENCE THEOREM

The volume integral of the divergence of a vector field over a volume is equal to the surface integral of the normal component of this vector over the surface bounding the volume.

$$\iiint_v \nabla \cdot A \, dv = \oiint_s A \cdot ds$$

Proof: The divergence of any vector A is given by

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Take the volume integral on both sides,

$$\iiint_v \nabla \cdot A \, dv = \iiint_v \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] dx \, dy \, dz$$

$$[\because dv = dx \, dy \, dz]$$

$$\iiint \frac{\partial A_x}{\partial x} dx \, dy \, dz = \iint \left[\int \frac{\partial A_x}{\partial x} \cdot dx \right] dy \, dz$$

But $\int_{x_1}^{x_2} \frac{\partial A_x}{\partial x} dx = A_{x_2} - A_{x_1} = A_x$

Then, $\iiint_v \frac{\partial A_x}{\partial x} dx \, dy \, dz = \iint_s A_x \, dy \, dz = \iint_s A_x \, ds_x$

where $dy \, dz = ds_x = x$ component of surface area ds.

Similarly, the following integrands become

$$\iiint_v \frac{\partial A_y}{\partial y} dx \, dy \, dz = \iint_s A_y \, ds_y$$

$$\iiint_v \frac{\partial A_z}{\partial z} dx \, dy \, dz = \iint_s A_z \, ds_z$$

Then,

$$\iiint \nabla \cdot A \, dv = \iiint_v \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right] dx \, dy \, dz$$

$$= \iint_s (A_x \, ds_x + A_y \, ds_y + A_z \, ds_z)$$

$$= \oiint_s A \cdot ds$$

$$\iiint \nabla \cdot A \, dv = \oiint_s A \cdot ds$$

The divergence theorem applies to time varying as well as static fields. This theorem is used to convert the volume integral of a divergence of a vector into a closed surface integral.

1.6. STOKE'S THEOREM

The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any closed surface.

$$\oint H \cdot dl = \iint_s \nabla \times H \, ds$$

Proof: Consider an arbitrary surface. This is broken up into incremental surfaces of area Δs as shown in Fig.1.9.

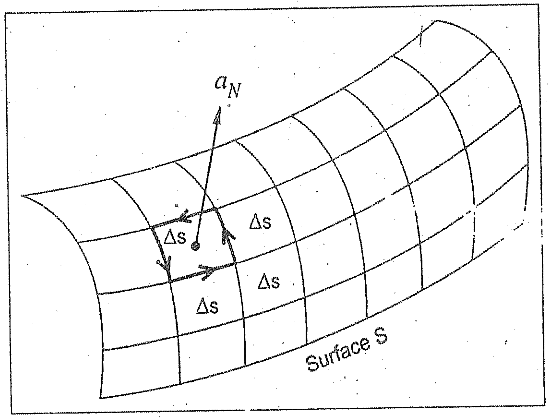


Fig. 1.9. Surface S consists of small surface Δs

If H is any field vector, then by definition of the curl to one of these incremental surfaces.

$$\frac{\oint H \cdot dl\, \Delta s}{\Delta s} = (\nabla \times H)\ a_N$$

where N indicates normal to the surface and $dl\ \Delta s$ indicate that the closed path of an incremental area Δs.

The curl of H normal to the surface can be written as

$$\frac{\oint H \cdot dl\, \Delta s}{\Delta s} = (\nabla \times H) \cdot a_N$$

or

$$\oint H \cdot dl\, \Delta s = (\nabla \times H) \cdot a_N\, \Delta s$$
$$= (\nabla \times H) \cdot \Delta s$$

where a_N is a unit vector normal to Δs.

The closed integral for whole surface s is given by the surface integral of the normal component of curl H.

$$\oint H \cdot dl = \iint_s \nabla \times H \cdot ds$$

This theorem is used to convert the surface integral of curl of a vector into a closed line integral.

1.7. COULOMB'S LAW

Charles Coulomb performed an elaborate series of experiments to determine quantitatively the force exerted between two objects each having a static charge.

Coulomb stated that

"The force between two very small objects separated by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them."

A small charged particle with negligible dimension is called a point charge. The coulomb's law refers to the force of interaction between two point charges. The force is repulsive when both the charges are like and attractive when the charges are unlike. The coulomb's law can be stated that,

The force of attraction or repulsion between any two point charges is directly proportional to the product of two charges and inversely proportional to the square of the distance between them.

Consider the two point charges Q_1 and Q_2 separated by a distance 'r'. The force of interaction between two point charges is given as follows:

$$F \propto \frac{Q_1 Q_2}{r^2}$$

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon r^2} \text{ Newtons}$$

$q_1 = q_2 = 1\,C$

$r = 1\,m$

$\therefore F = 9 \times 10^9 N$

one coulomb.

where ε is permittivity of the medium or dielectric constant which is written as

$$\varepsilon = \varepsilon_0 \varepsilon_r \text{ Farads/metre}$$

$\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$

Relative permittivity of the medium, $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$

$$\varepsilon_r = \frac{\text{Permittivity of the medium}}{\text{Permittivity of the free space (or) vacuum}}$$

$$\left.\begin{array}{r}\text{Permittivity of the free}\\ \text{space or vacuum}\end{array}\right\} \varepsilon_0 = \frac{1}{36\pi \times 10^9} = 8.854 \times 10^{-12} \text{ F/m}$$

$C^2 N^{-1} m^{-2}$

Coulomb's Law in Vector Form

Consider two charges Q_1 and Q_2 at distant vectors r_1 and r_2 from the origin respectively as shown in Fig.1.10. r_{12} represents the distant vector from Q_1 to Q_2.

$$r_{12} = r_2 - r_1$$

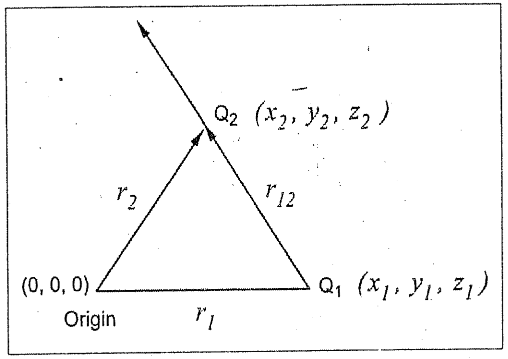


Fig. 1.10. Force between two charges Q_1 and Q_2

The vector F is the force between Q_1 and Q_2.

The vector form of Coulomb's law is

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r_{12}^2} \times \bar{a}_{12}$$

where $\bar{a}_{12}$ is the unit vector in the direction of r_{12}.

$$\bar{a}_{12} = \frac{\bar{r}_2 - \bar{r}_1}{|r_2 - r_1|}$$

$$F = \frac{Q_1 Q_2 (\bar{r}_2 - \bar{r}_1)}{4\pi \varepsilon_0 (r_2 - r_1)^2 (r_2 - r_1)}$$

$$\therefore \quad F = \frac{Q_1 Q_2 [(x_2 - x_1)\bar{a}_x + (y_2 - y_1)\bar{a}_y + (z_2 - z_1)\bar{a}_z]}{4\pi \varepsilon_0 [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}}$$

The force is attractive if the charges are of opposite sign and is repulsive if the charges are alike in sign.

Principle of Superposition

Consider a system consists of n point charges $Q_1, Q_2, \ldots\ldots Q_n$ at distances $r_1, r_2, \ldots\ldots r_n$ respectively. The force on n^{th} charge is given by vector sum of all the individual forces. This principle is called principle of superposition.

The force F_i on the i^{th} charge can be written by Coulomb's law,

$$F = \sum_{j=1}^{n} \frac{Q_i Q_j}{4\pi \varepsilon_0 r_{ji}^2} \times a_{ij} \qquad [i \neq j]$$

where Q_i is the i^{th} charge.

Q_j is any of the charges other than Q_i.

r_{ji} is the distance between the j^{th} and i^{th} charges.

1.8. CONTINUOUS CHARGE DISTRIBUTION

If the charge are distributed instead of concentrated at one point, it is better to define charge distribution in terms of charge density. The different types of charge density are given below.

Linear Charge Density

It is defined as the total charge distributed over a line or curve.

$$\rho_l = \underset{\Delta l \to 0}{\mathrm{Lt}} \left(\frac{\Delta Q}{\Delta l} \right)$$

This gives the total charge per length. It is given by

$$\rho_l = \frac{Q}{l} \quad \text{Coulomb/metre (c/m)}$$

Surface Charge Density

It is defined as the total charge distributed over a surface.

$$\rho_s = \underset{\Delta s \to 0}{\mathrm{Lt}} \left(\frac{\Delta Q}{\Delta s} \right)$$

This gives the total charge per area. It is given by

$$\rho_s = \frac{Q}{s} = \frac{Q}{A} \quad \text{Coulomb/sq.metre } (c/m^2)$$

Volume Charge Density

It is defined as the total charge distributed over a volume.

$$\rho_v = \underset{\Delta v \to 0}{\mathrm{Lt}} \left(\frac{\Delta Q}{\Delta v} \right)$$

This gives the total charge per volume. It is given by

$$\rho_v = \frac{Q}{v} \quad \text{Coulomb/cubic metre } (c/m^3)$$

1.9. ELECTRIC FIELD INTENSITY

Consider a charge Q fixed in position and move another charge q, call it as test charge, around the fixed charge. The test charge experiences a force around the fixed charge Q *i.e.*, electric field is set up around the Q and any charge brought into this field, will experiences a force.

The ***electric field*** or ***electric field intensity*** is defined as the electric force per unit charge. It is given by

$$E = \frac{F}{q}$$

According to Coulomb's law,

$$F = \frac{Qq}{4\pi\varepsilon r^2}$$

$$\text{Electric field } E = \frac{F}{q} = \frac{Q}{4\pi\varepsilon r^2} \text{ Newtons/Coulomb}$$

The another unit of electric field is volts/metre.

$$E = \frac{Q}{4\pi\varepsilon r^2} \text{ V/m}$$

1.9.1. Electric Field due to Charge Distribution

Consider a continuous charge distribution (may be line, surface or volume). To determine the electric field due to charge distribution at any point P, divide this into n small sections, which approach zero in the limit. Each section contains a charge dQ_i. The electric field intensity is obtained in the limit of sum as

$$E = \underset{n\to\infty}{\text{Lt}} \sum_{i=1}^{n} \frac{\Delta Q_i}{4\pi\varepsilon_0 r^2}$$

where r is the position vector of the point P.

For line charge $\Delta Q_i = \rho_l \, dl$, the electric field intensity

$$E = \frac{1}{4\pi\varepsilon_0} \int_l \frac{\rho_l}{r^2} dl$$

For surface charge $\Delta Q_i = \rho_s \, ds$, the electric field intensity

$$E = \frac{1}{4\pi\varepsilon_0} \int_s \frac{\rho_s \, ds}{r^2}$$

For volume charge, $\Delta Q_i = \rho_v \, dv$, the electric field intensity

$$E = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho_v \, dv}{r^2}$$

1.9.2. Electric Field due to Line Charge Distribution

Consider a uniformly charged line of length 'L' whose linear charge density is ρ_l coulomb/metre. Consider a small element dl at a distance 'l' from one end of the charged line as shown in Fig.1.11. Let P be any point at a distance 'r' from the element dl.

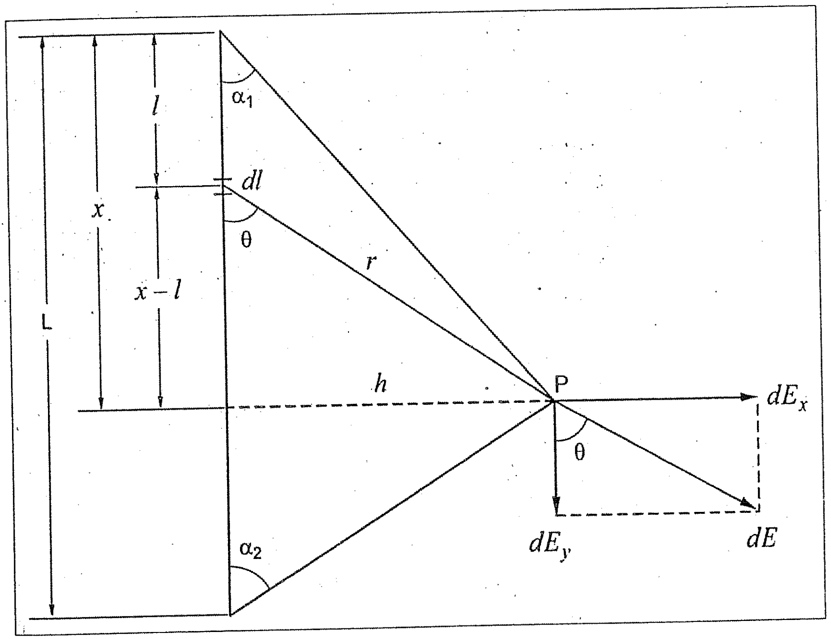


Fig. 1.11. Line charge

The electric field at a point P due to the charge element $\rho_l \, dl$ is given by

$$dE = \frac{\rho_l \, dl}{4\pi \varepsilon r^2}$$

The x and y components of electric field dE are given by

$$dE_x = dE \sin\theta$$

$$dE_y = dE \cos\theta$$

Then $$dE_x = \frac{\rho_l \, dl \sin\theta}{4\pi \varepsilon r^2}$$

From the Fig.1.11, $$x - l = h \cot\theta$$

$$-dl = -h \operatorname{cosec}^2\theta \cdot d\theta$$

and $$\frac{h}{r} = \sin\theta$$

$$r = h \operatorname{cosec}\theta$$

Substituting the values of dl and r in dE_x equation

$$dE_x = \frac{\rho_l (h \operatorname{cosec}^2\theta \, d\theta) \sin\theta}{4\pi \varepsilon (h \operatorname{cosec}\theta)^2}$$

$$dE_x = \frac{\rho_l \sin\theta \, d\theta}{4\pi \varepsilon h}$$

The electric field E_x due to the entire length of line charge is given by

$$E_x = \int_{\alpha_1}^{\pi-\alpha_2} \frac{\rho_l \sin\theta \, d\theta}{4\pi \varepsilon h}$$

$$E_x = \frac{\rho_l}{4\pi \varepsilon h} \int_{\alpha_1}^{\pi-\alpha_2} \sin\theta \, d\theta$$

$$E_x = \frac{\rho_l}{4\pi \varepsilon h} \left[-\cos\theta\right]_{\alpha_1}^{\pi-\alpha_2}$$

$$E_x = \frac{\rho_l}{4\pi \varepsilon h} [\cos\alpha_1 + \cos\alpha_2]$$

Similarly for y component of E,

$$dE_y = \frac{\rho_l}{4\pi \varepsilon h} \cos\theta \, d\theta$$

$$E_y = \frac{\rho_l}{4\pi \varepsilon h} \int_{\alpha_1}^{\pi-\alpha_2} (\cos\theta) \, d\theta$$

$$E_y = \frac{\rho_l}{4\pi \varepsilon h} \left[\sin\theta\right]_{\alpha_1}^{\pi-\alpha_2}$$

$$E_y = \frac{\rho_l}{4\pi \varepsilon h} [\sin\alpha_2 - \sin\alpha_1]$$

Case 1: If the point P is at bisector of a line, then $\alpha_1 = \alpha_2 = \alpha$.

$E_y = 0$, E becomes E_x

$$E = \frac{\rho_l}{2\pi \varepsilon h}$$

Case 2: If the line is infinitely long, then $\alpha = 0$

$E_y = 0$, E becomes E_x

$$E = \frac{\rho_l}{2\pi \varepsilon h}$$

1.9.3. Electric Field due to a Charged Circular Disc

Consider a circular disc of radius 'R' is charged uniformly with a charge density of ρ_s coulomb/m^2. Let P be any point on the axis of the disc at a distance from the centre. Consider an annular ring of radius 'r' and of radial thickness dr as shown in Fig.1.12. The area of the annular ring is $ds = 2\pi r \, dr$. The field intensity at point P due to the charged annular ring is given by

$$dE = \frac{\rho_s \, ds}{4\pi \varepsilon d^2}$$

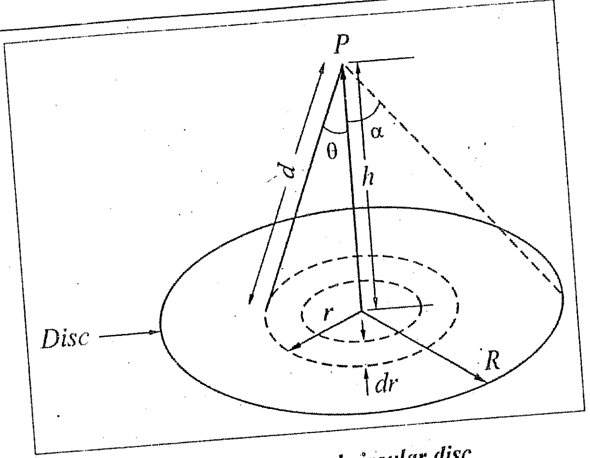


Fig. 1.12. Charged circular disc

Since the horizontal component of electric field intensity is zero, the vertical component is given by

$$dE_y = \frac{\rho_s \cdot ds}{4\pi\,\varepsilon\, d^2}\cos\theta$$

$$dE_y = \frac{\rho_s \cdot 2\pi\, r\, dr}{4\pi\,\varepsilon\, d^2}\cos\theta$$

From the Fig.1.12,

$$r = h\tan\theta\,; \qquad dr = h\sec^2\theta\cdot d\theta$$

$$d = \frac{r}{\sin\theta}$$

Substituting the values of dr and d in dE_y equation

then

$$dE_y = \frac{\rho_s\, 2\pi\, r\, h\sec^2\theta\cdot d\theta\cos\theta\sin^2\theta}{4\pi\,\varepsilon\, r^2}$$

$$= \frac{\rho_s\, h\sec\theta\cdot\sin^2\theta\, d\theta}{2\,\varepsilon\, r}$$

$$= \frac{\rho_s \sec\theta\sin^2\theta\, d\theta}{2\,\varepsilon\tan\theta} \qquad \left[\because \frac{r}{h} = \tan\theta\right.$$

$$= \frac{\rho_s}{2\,\varepsilon}\sin\theta\cdot d\theta \qquad [\because \sec\theta\sin\theta = \tan\theta$$

The total electric field due to the charged disc

$$E = \int_{\theta=0}^{\alpha} dE_y = \frac{\rho_s}{2\,\varepsilon}\int_0^{\alpha}\sin\theta\cdot d\theta = \frac{\rho_s}{2\,\varepsilon}(1-\cos\theta)$$

$$E = \frac{\rho_s}{2\,\varepsilon}\left[1 - \frac{h}{\sqrt{h^2+R^2}}\right]$$

1.9.4. Electric Field due to a Infinite Uniformly Charged Sheet

Consider an infinite plane sheet which is uniformly charged with a charge density of ρ_s coulomb/m^2 as shown in Fig.1.13.

The field intensity at any point 'P' due to infinite plane sheet of charge can be evaluated by applying expression of charged circular disc.

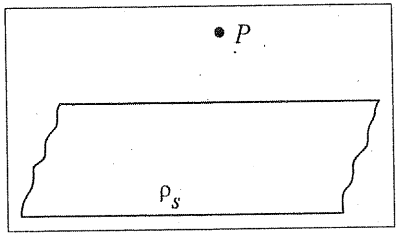


Fig. 1.13. Infinite plane sheet of charge

$$E = \frac{\rho_s}{2\,\varepsilon}(1-\cos\alpha)$$

$$E = \frac{\rho_s}{2\,\varepsilon} \qquad [\because \alpha = 90°]$$

1.9.5. Electric Field between Two Infinite Uniformly Charged Sheets

Consider two infinite plane sheets are separated by a distance of 'd'. The upper sheet is uniformly distributed with charge density of ρ_s coulomb/m^2 and the lower sheet is uniformly distributed with charged density of $-\rho_s$ coulomb/m^2. Consider any point P between the planes as shown in Fig.1.14.

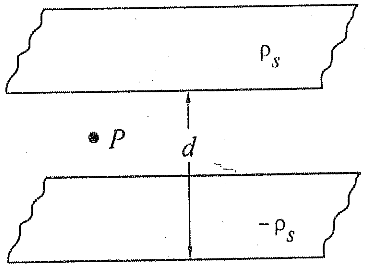


Fig. 1.14. Two infinite plane sheets of charged

The electric field intensity at any point P due to the two oppositely charged infinite plane is additive. It is given by

$$E = \frac{\rho_s}{2\,\varepsilon} + \frac{\rho_s}{2\,\varepsilon}$$

$$E = \frac{\rho_s}{\varepsilon}$$

1.10. GAUSS'S LAW

note

Electric Flux (χ)

If a test charge is moved towards the charge Q, the test charge will experience force due to the main charge Q. The lines of force can be designated as electric flux which is equal to the charge itself. The electric flux χ eminates from electric charge Q.

$$\chi = Q$$

Electric Flux Density (D)

Electric flux density or Displacement density is defined as the electric flux per unit area.

$$D = \frac{Q}{A} \text{ Coulomb/metre}^2$$

For a sphere surface area,

$$A = 4\pi\, r^2$$

$$D = \frac{Q}{4\pi\, r^2}$$

$$\text{But} \quad E = \frac{Q}{4\pi\,\varepsilon\, r^2}$$

$$\therefore \quad D = \varepsilon\, E$$

1.10.1. Gauss's Law

Faraday's experiments lead to the following statement, which is known as Gauss's law.

The electric flux passing through any closed surface is equal to the total charge enclosed by the surface.

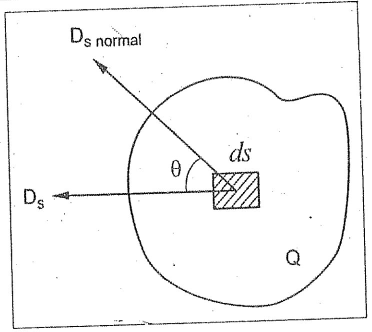


Fig. 1.15. Closed surface having charge Q

Consider a small element of area ds in a plane surface having a charge Q and P be a point in the element. At every point on the surface, the electric flux density D will have value D_s. Let D_s make an angle θ with ds as shown in Fig.1.15. The flux crossing ds is the product of the normal component of D_s and ds.

$$d\chi = D_{s\,\text{normal}}\ ds$$

$$= D_s \cos\theta\ ds$$

$$d\chi = D_s \cdot ds \qquad \text{[dot product]}$$

The total flux passing through the closed surface is given by

$$\chi = \int d\chi = \oint_s D_s \cdot ds$$

$$\chi = Q$$

In terms of volume charge density

$$\chi = \int_v \rho_v \cdot dv = Q$$

Proof: Consider a charge Q at the origin of a spherical co-ordinate system, whose co-ordinates are r, θ, ϕ as shown in Fig.1.16. The radius of the sphere is r.

The electric field intensity due to the charge Q is

$$E = \frac{Q}{4\pi\,\varepsilon\, r^2}$$

The electric flux density $$D = \varepsilon E$$

$$D = \frac{Q}{4\pi r^2}$$

Consider a small element of area ds on the surface of the sphere at a distance r from the origin as shown in Fig.1.16.

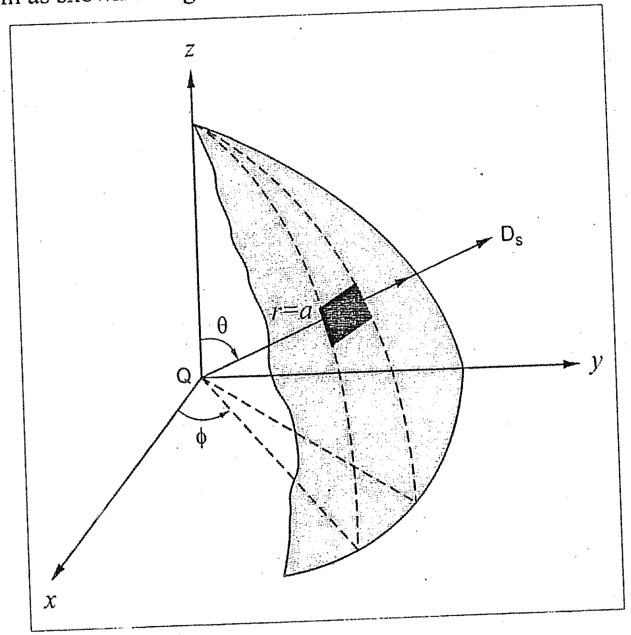


Fig. 1.16. Surface element ds in the spherical co-ordinate

$$ds = r \cdot d\theta \cdot r \sin\theta \cdot d\phi$$

$$= r^2 \cdot \sin\theta \cdot d\theta \cdot d\phi$$

The electric flux $$\chi = \int d\chi = \oint_s D \cdot ds$$

$$= \int_s \frac{Q}{4\pi r^2} r^2 \cdot \sin\theta \, d\theta \cdot d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta \cdot d\theta \cdot d\phi$$

$$= \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} \left[\int_{\theta=0}^{\pi} \sin\theta \cdot d\theta \right] d\phi$$

$$= \frac{Q}{4\pi} \int_{0}^{2\pi} [-\cos\theta]_0^{\pi} \, d\phi$$

$$= \frac{Q}{4\pi} \int_{0}^{2\pi} 2 \cdot d\phi$$

$$= \frac{Q}{4\pi} \left[\phi \right]_0^{2\pi}$$

$$\therefore \quad \chi = Q$$

The electric flux crossing the surface is equal to the charge enclosed by the surface.

The Gauss's law can be stated in the point form as follows.

The divergence of electric flux density is equal to the volume charge density.

$$\nabla \cdot D = \rho_v$$

1.11. APPLICATIONS OF GAUSS'S LAW

1.11.1. Gaussian Surface

Gauss's law is applied to the surface if the following conditions are satisfied.

(*i*) The surface is closed.

(*ii*) Electric flux density D is either normal or tangential to the surface at each point of the surface.

(*iii*) Electric flux density D is constant over the part of the surface where D is normal.

The surface over which Gauss's law is applied is called ***Gaussian surface.*** Gauss's law is used to determine the charge (Q) enclosed or flux density (D) or electric field intensity (E) for symmetrical charge distribution.

1.11.2. Infinite Line Charge

Consider an infinite line charge of ρ_l c/m as the Z-axis of cylindrical co-ordinate. Consider a circular cylinder of radius 'r' with length 'l' as Gaussian surface as shown in Fig.1.17. The electric flux density (D_s) is normal to the surface of the cylinder at everywhere.

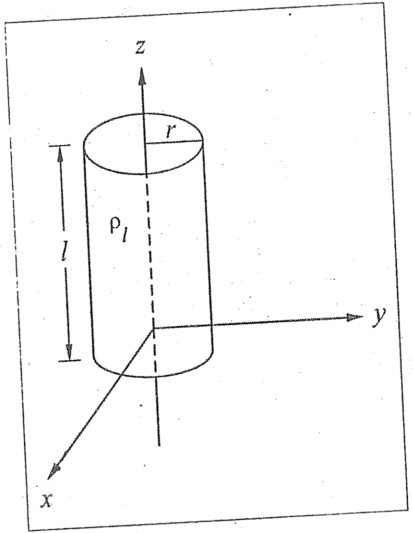


Fig. 1.17. Circular cylinder in cylindrical co-ordinate system

By applying Gauss's law to the closed surface (Gaussian surface) of the cylinder

$$Q = \oint D_s \cdot ds$$

$$= D_s \int_{\text{sides}} ds + D_s \int_{\text{top}} ds + D_s \int_{\text{bottom}} ds$$

Electric flux on top and bottom of cylinder (except surface of the cylinder) is zero.

Then, $$Q = D_s \int_{\text{sides}} ds = D_s \int_{z=0}^{l} \int_{\phi=0}^{2\pi} r \, d\phi \, dz$$

$$= D_s \int_{z=0}^{l} r \, [\phi]_0^{2\pi} \, dz$$

$$= D_s \int_{z=0}^{l} 2\pi r \cdot dz$$

$$Q = D_s \cdot 2\pi r \, l$$

$$\therefore \; D_s = \frac{Q}{2\pi r \, l}$$

If $\rho_l = \dfrac{Q}{l}$ is the linear charge density.

$$D_s = \frac{\rho_l}{2\pi r}$$

Electric field intensity $E = \dfrac{D_s}{\varepsilon_0} = \dfrac{\rho_l}{2\pi \varepsilon_0 r}$

$$E = \frac{\rho_l}{2\pi \varepsilon_0 r}$$

1.11.3. Single Shell of Charge

Let a positive charge Q is uniformly distributed over a spherical surface of radius as shown in the Fig.1.18.

By applying Gauss's law inside the shell the integral of flux density D over a spherical surface (Gaussian surface) is zero as no charge is enclosed by the surface.

$$\oint D \cdot ds = 0 \qquad r < a$$

$$\varepsilon \oint E \cdot ds = 0$$

$$E = 0 \qquad r < a$$

Electric field is zero inside the shell.

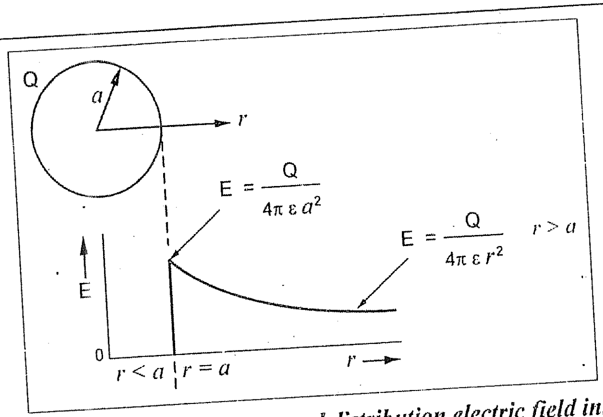


Fig. 1.18. Single shell of charge and distribution electric field intensity

By applying Gauss's law just outside the shell, the integral of flux density D over a spherical surface is the charge of the shell.

$$\oint_s D \cdot ds = Q$$

$$\varepsilon \oint_s E \cdot ds = Q$$

$$\oint_s E \cdot ds = \frac{Q}{\varepsilon}$$

$$E \cdot (4\pi r^2) = \frac{Q}{\varepsilon} \qquad r > a$$

$$E = \frac{Q}{4\pi \varepsilon r^2}$$

This is the electric field just outside the spherical shell.

1.11.4. Two Concentric Shells of Charge

(a) Electric field intensity between two shells

Consider two spherical shells of radii a and b. Let Q_1 and Q_2 be the charges uniformly distributed over the inner shell of radius 'a' and outer shell of radius 'b' respectively.

By applying Gauss's law, the line integral of flux density D over a closed surface is zero.

$$\oint D \cdot ds = 0 \qquad [\because D = \varepsilon E]$$

$$\therefore \; E = 0 \qquad r < a$$

The field intensity between the two shells ($a < r < b$)

$$E = \frac{Q_1}{4\pi \varepsilon r^2} \qquad a < r < b$$

The field intensity just outside both the shells due to Q_1 and Q_2

$$E = \frac{Q_1 + Q_2}{4\pi \varepsilon r^2} \qquad r > b > a$$

At the inner shell $r = a$, the electric field intensity

$$E = \frac{Q_1}{4\pi \varepsilon r^2}$$

At the outer shell $r = b$, the electric field intensity

$$E = \frac{Q_1 + Q_2}{4\pi \varepsilon r^2}$$

The variation of electric field intensity is as shown in Fig.1.19.

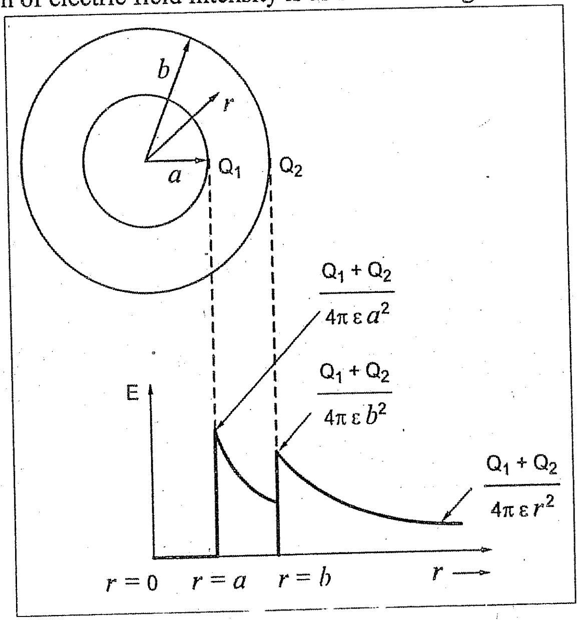


Fig. 1.19.

1.11.5. Co-Axial Cylinders

Consider two co-axial cylinders of inner radius 'a' and outer radius 'b'. The two cylinder are uniformly distributed with charge density of ρ_l coulomb/metre. Let the inner cylinder has charge density of ρ_l and outer cylinder has $-\rho_l$ as shown in Fig.1.20. Let E be the electric field at any distance r from the axis of cylinders of length l. The elemental area $ds = 2\pi r l$.

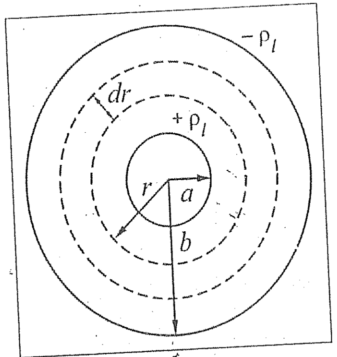


Fig. 1.20. Cross-section of co-axial cylinders

By applying Gauss's law

$$\oint D \cdot ds = \rho_l l$$

$$\varepsilon \oint E \cdot ds = \rho_l l \qquad [\because D = \varepsilon E]$$

$$E \cdot 2\pi r l = \frac{\rho_l \cdot l}{\varepsilon} \qquad [\because \int ds = 2\pi r l]$$

$$E = \frac{\rho_l}{2\pi \varepsilon r}$$

SOLVED EXAMPLES

✓ **Example 1.1** *Given that $A = \bar{a}_x + \bar{a}_y$, $B = \bar{a}_x + 2\,\bar{a}_z$ and $C = 2\,\bar{a}_y + \bar{a}_z$, find $\bar{A} \cdot (\bar{B} \times \bar{C})$ and $(\bar{A} \times \bar{B}) \cdot \bar{C}$.*

☺***Solution:***

$$\bar{B} \times \bar{C} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -4\,\bar{a}_x - \bar{a}_y + 2\,\bar{a}_z$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{a}_x + \bar{a}_y) \cdot (-4\,\bar{a}_x - \bar{a}_y + 2\,\bar{a}_z)$$

$$= -4 - 1 + 0 = -5$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\,\bar{a}_x - 2\,\bar{a}_y - \bar{a}_z$$

$$(\bar{A} \times \bar{B}) \cdot \bar{C} = (2\,\bar{a}_x - 2\,\bar{a}_y - \bar{a}_z) \cdot (2\,\bar{a}_y + \bar{a}_z)$$

$$= 0 - 4 - 1 = -5$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C}$$

☺☺

✓ **Example 1.2** *Show that the vectors $\bar{A} = 4\,\bar{a}_x - 2\,\bar{a}_y + 2\,\bar{a}_z$ and $\bar{B} = -6\,\bar{a}_x + 3\,\bar{a}_y - 3\,\bar{a}_z$ are parallel to each other.*

☺***Solution:***

$$\bar{A} = 4\,\bar{a}_x - 2\,\bar{a}_y + 2\,\bar{a}_z$$

$$\bar{B} = -6\,\bar{a}_x + 3\,\bar{a}_y - 3\,\bar{a}_z$$

The condition for two vectors A and B to be parallel is given by

$$\bar{A} \times \bar{B} = 0$$

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ 4 & -2 & 2 \\ -6 & 3 & -3 \end{vmatrix}$$

$$= \bar{a}_x[6-6] - \bar{a}_y[-12+12] + \bar{a}_z[12-12]$$

$$= 0$$

$\therefore$ Vectors $\bar{A}$ and $\bar{B}$ are parallel to each other.

☺☺

Example 1.3 *Show that the vectors* $\bar{A} = 6\bar{a}_x + 2\bar{a}_y - 5\bar{a}_z$ *and* $\bar{B} = 5\bar{a}_x - 5\bar{a}_y + 4\bar{a}_z$ *are perpendicular to each other.*

☺Solution:

$$\bar{A} = 6\bar{a}_x + 2\bar{a}_y - 5\bar{a}_z$$

$$\bar{B} = 5\bar{a}_x - 5\bar{a}_y + 4\bar{a}_z$$

The condition for two vectors A and B to be perpendicular is given by

$$\mathbf{\bar{A} \cdot \bar{B} = 0}$$

$$\bar{A} \cdot \bar{B} = 6 \times 5 - 2 \times 5 - 5 \times 4 = 0$$

☺☺

∴ Vector $\bar{A}$ and $\bar{B}$ are perpendicular to each other.

Example 1.4 *Two vectorial quantities* $\vec{A} = 4\hat{i} + 3\hat{j} + 5\hat{k}$ *and* $\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$ *are known to be oriented in two unique directions. Determine the angular separation between them.*

☺Solution:

$$\vec{A} = 4\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|} \text{ from definition of Dot product}$$

$$\cos\theta = \frac{4-6+10}{\sqrt{16+9+25}\ \sqrt{1+4+4}} = \frac{8}{\sqrt{450}}$$

Angular separation between the two vectorial quantities

$$\theta = \cos^{-1}(0.377) = 67.844°$$

☺☺

Example 1.5 *Given* $\vec{A} = 5\vec{a}_x$ *and* $B = 4\vec{a}_x + t\,\vec{a}_y$, *find 't' such that the angle between* $\vec{A}$ *and* $\vec{B}$ *at* 45°.

☺Solution:

$$\vec{A} = 5\vec{a}_x$$

$$B = 4\vec{a}_x + t\,\vec{a}_y$$

$$\theta = 45°$$

$$\cos\theta = \frac{A \cdot B}{|A||B|}$$

$$\cos 45° = \frac{5\vec{a}_x\ (4\vec{a}_x + t\,\vec{a}_y)}{|5\vec{a}_x||4\vec{a}_x + t\,\vec{a}_y|}$$

$$\frac{1}{\sqrt{2}} = \frac{20+0}{\sqrt{5^2}\ \sqrt{16+t^2}}$$

$$\frac{1}{2} = \frac{20^2}{25\,(16+t^2)}$$

$$25\,(16+t^2) = 2\,(400)$$

$$16+t^2 = 32$$

$$t^2 = 16$$

$$\therefore t = 4$$

☺☺

Example 1.6 *Find the divergence of the vector function.*

$$A = x^2\bar{a}_x + (xy)^2\bar{a}_y + 24x^2y^2z^3\bar{a}_z$$

☺Solution:

$$\nabla \cdot A = \left(\bar{a}_x\frac{\partial}{\partial x} + \bar{a}_y\frac{\partial}{\partial y} + \bar{a}_z\frac{\partial}{\partial z}\right)[x^2\bar{a}_x + (xy)^2\bar{a}_y + 24x^2y^2z^3\bar{a}_z$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(x^2y^2) + \frac{\partial}{\partial z}(24x^2y^2z^3)$$

$$= \mathbf{2x + 2x^2y + 72x^2y^2z^2}$$

☺☺

Example 1.7 *If* $F = x^2\bar{a}_x + y^2\bar{a}_y + z^2\bar{a}_z$, *then find divergence of F and url of F.*

☺Solution:

$$\text{Div } F = \nabla \cdot F = \left(\bar{a}_x\frac{\partial}{\partial x} + \bar{a}_y\frac{\partial}{\partial y} + \bar{a}_z\frac{\partial}{\partial z}\right)(x^2\bar{a}_x + y^2\bar{a}_y + z^2\bar{a}_z)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z$$

$$\nabla \cdot F = 2(x + y + z)$$

$$\text{Curl. F} = \nabla \times F = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \bar{a}_y\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right]$$

$$+ \bar{a}_z\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \bar{a}_x \cdot 0 + \bar{a}_y \cdot 0 + \bar{a}_z \cdot 0 = 0$$

☺☺

Example 1.8 *If $\bar{r} = 5x\bar{i} + 2y\bar{j} - 3z\bar{k}$, find div $\bar{r}$.*

☺***Solution:***

Given: $\bar{r} = 5x\bar{i} + 2y\bar{j} - 3z\bar{k}$

From the definition of divergence of a vector,

$$\text{div } \bar{r} = \nabla \cdot \bar{r}$$

$$= \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right](5x\bar{i} + 2y\bar{j} - 3z\bar{k})$$

$$= \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(2y) - \frac{\partial}{\partial z}(3z)$$

$$= 5 + 2 - 3 = 4$$

∴ Divergence of given function $\bar{r}$ is 4. ☺☺

Example 1.9 *Evaluate the divergence of function $V = xy\bar{i} + 2yz\bar{j} + 3zx\bar{k}$ at point (–1, –1, 2).*

☺***Solution:***

Given: $V = xy\,\bar{i} + 2yz\,\bar{j} + 3zx\,\bar{k}$, **point P(–1, –1, 2)**

From the definition of divergence,

$$\nabla \cdot V = \left[\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right](xy\,\bar{i} + 2yz\,\bar{j} + 3zx\,\bar{k})$$

$$= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(3zx)$$

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$$\nabla \cdot V = y + 2z + 3x$$

Divergence at (–1, –1, 2) is

$$\nabla \cdot V\Big|_{(-1,-1,2)} = (-1) + 2(2) + 3(-1) = -1 + 4 - 3 = 0$$

∴ Divergence of function V at (– 1, – 1, 2) is zero. ☺☺

Example 1.10 *Find the nature of the field given by determining the divergence and curl $F_1 = 30\bar{I}_x + 2xy\bar{I}_y + 5xz^2\bar{I}_z$.*

☺***Solution:*** $F_1 = 30\bar{I}_x + 2xy\bar{I}_y + 5xz^2\bar{I}_z$

$$\text{Div } F_1 = \nabla \cdot F_1$$

$$= \frac{\partial}{\partial x}(30) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(5xz^2)$$

$$= 0 + 2x + 10xz$$

$$\nabla \cdot F_1 = 2x + 10xz$$

Divergence exist. So field is non-solenoidal.

$$\text{curl } F_1 = \nabla \times F_1$$

$$\nabla \times F_1 = \begin{vmatrix} \bar{I}_x & \bar{I}_y & \bar{I}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix}$$

$$= \bar{I}_x(0) - \bar{I}_y\left(\frac{\partial}{\partial x}(5xz^2) - 0\right) + \bar{I}_z\left(\frac{\partial}{\partial x}(2xy) - 0\right)$$

$$\nabla \times F_1 = -5z^2\,\bar{I}_y + 2y\,\bar{I}_z$$ ☺☺

Curl exist. So field is rotational.

Example 1.11 *Determine the divergence of these vector fields.*

(i) $P = x^2yz\ \bar{a}_x + xz\,\bar{a}_z$

(ii) $Q = \rho \sin\phi\ \bar{a}_\rho + \rho^2 z\,\bar{a}_\phi + z\cos\phi\ \bar{a}_z$

(iii) $T = \left(\frac{1}{r^2}\right)\cos\theta\ \bar{a}_r + r\sin\theta\cos\phi\ \bar{a}_\theta + \cos\theta\ \bar{a}_\theta$

☺***Solution:***

(i) $$\nabla.P = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z} \quad \text{(Rectangular)}$$

$$= \frac{\partial}{\partial x}(x^2yz) + 0 + \frac{\partial}{\partial z}(xz)$$

$$= \mathbf{2xyz + x}$$

(ii) $$\nabla.Q = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_z}{\partial z} \quad \text{(Cylindrical)}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$= \frac{1}{\rho}\cdot 2\rho\sin\phi + 0 + \cos\phi$$

$$= \mathbf{2\sin\phi + \cos\phi}$$

(iii) $$\nabla.T = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, T_\theta) + \frac{1}{r\sin\theta}\frac{\partial T_\phi}{\partial \phi} \quad \text{(Spherical)}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\cdot r\sin\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$$

$$= 0 + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\cos\phi\sin^2\theta) + 0$$

$$= \cos\phi\,\frac{\partial}{\partial \theta}(\sin^2\theta) = \cos\phi\cdot 2\cdot\sin\theta\cdot\cos\theta$$

$$= \mathbf{2\cos\phi\sin\theta\cdot\cos\theta}$$ ☺☺

Example 1.12 *Find curl $\vec{H}$, if $\vec{H} = (2\rho\cos\phi\,\vec{a}_\rho - 4\rho\sin\phi\,\vec{a}_\phi + 3\,\vec{a}_z)$.*

☺***Solution:***

$$\nabla \times H = \begin{vmatrix} \dfrac{1}{\rho}\vec{a}_\rho & \vec{a}_\phi & \dfrac{1}{\rho}\vec{a}_y \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ \vec{H}_\rho & \rho H_\phi & H_z \end{vmatrix}$$

$$= \vec{a}_\rho\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right) + \vec{a}_\phi\left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right) + \vec{a}_z\,\frac{1}{\rho}\left(\frac{\partial \rho H_\phi}{\partial \phi} - \frac{\partial H_\rho}{\partial \phi}\right)$$

$$\vec{H} = 2\rho\cos\phi\,\vec{a}_\rho - 4\rho\sin\phi\,\vec{a}_\phi + 3\,\vec{a}_z$$

$$\nabla \times H = \vec{a}_z\,\frac{1}{\rho}\,[4\rho\sin\phi\,\vec{a}_\phi + 2\rho\sin\phi\,\vec{a}_\rho]$$

$$= \mathbf{4\sin\phi\,\vec{a}_\phi\,\vec{a}_z + 2\sin\phi\,\vec{a}_\rho\ \vec{a}_z}$$ ☺☺

Example 1.13 *Show that the vector* $\bar{H} = 3y^4 z^2 \bar{a}_x + 4x^3 z^2 \bar{a}_y + 3x^2 y^2 \bar{a}_z$ *is solenoidal.*

☺**Solution:**

$$\nabla \cdot \bar{H} = \left(\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}\right)(3y^4 z^2 \bar{a}_x + 4x^3 z^2 \bar{a}_y + 3x^2 y^2 \bar{a}_z)$$

$$= \frac{\partial}{\partial x}(3y^4 z^2) + \frac{\partial}{\partial y}(4x^3 z^2) + \frac{\partial}{\partial z}(3x^2 y^2)$$

$$= 0 + 0 + 0 = 0$$

☺☺

$\nabla \cdot \bar{H} = 0.$ Hence $\bar{H}$ is solenoidal.

Example 1.14 *Show that the vector* $2xy\,\bar{a}_x + (x^2 + 2yz)\,\bar{a}_y + (y^2 + 1)\,\bar{a}_z$ *is irrotational.*

☺**Solution:** Let $\bar{F} = 2xy\,\bar{a}_x + (x^2 + 2yz)\,\bar{a}_y + (y^2 + 1)\,\bar{a}_z$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + 2yz & y^2 + 1 \end{vmatrix}$$

$$= \bar{a}_x\left(\frac{\partial}{\partial y}(y^2+1) - \frac{\partial}{\partial z}(x^2 + 2yz)\right) - \bar{a}_y\left(\frac{\partial}{\partial x}(y^2+1) - \frac{\partial}{\partial z}(2xy)\right)$$

$$+ \bar{a}_z\left(\frac{\partial}{\partial x}(x^2 + 2yz) - \frac{\partial}{\partial y}(2xy)\right)$$

$$= \bar{a}_x(2y - 2y) - \bar{a}_y(0 - 0) + \bar{a}_z(2x - 2x) = 0$$

☺☺

$\nabla \times \bar{F} = 0.$ Hence F is irrotational.

Example 1.15 *Show that the vector* $E = (6xy + z^3)\,\bar{a}_x + (3x^2 - z)\bar{a}_y + (3xz^2 - y)\,\bar{a}_z$ *is irrotational and find its scalar potential.*

☺**Solution:** Vector E is irrotational if $\nabla \times E = 0$.

$$\nabla \times E = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right] - \bar{a}_y\left[\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right]$$

$$+ \bar{a}_z\left[\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^2)\right]$$

$$= \bar{a}_x \cdot [-1 + 1] - \bar{a}_y \cdot [3z^2 - 3z^2] + \bar{a}_z \cdot [6x - 6x]$$

$$= 0$$

Hence E is irrotational.

$$-\nabla \cdot V = E = (6xy + z^3)\,\bar{a}_x + (3x^2 - z)\,\bar{a}_y + (3xz^2 - y)\,\bar{a}_z$$

$$-\left[\bar{a}_x \frac{\partial v}{\partial x} + \bar{a}_y \frac{\partial v}{\partial y} + \bar{a}_z \frac{\partial v}{\partial z}\right] = (6xy + z^3)\,\bar{a}_x + (3x^2 - z)\,\bar{a}_y + (3xz^2 - y)\,\bar{a}_z$$

Equating on both sides,

$$-\frac{\partial v}{\partial x} = 6xy + z^3$$

$$-\frac{\partial v}{\partial y} = 3x^2 - z$$

$$-\frac{\partial v}{\partial z} = 3xz^2 - y$$

Then,

$$-\partial v = (6xy + z^3)\,\partial x$$

$$-v = \int (6xy + z^3)\,\partial x$$

$$= 3x^2 y + xz^3 + c_1$$ where c_1 is constant.

$$-\partial v = (3x^2 - z)\,\partial y$$
$$-v = 3x^2 y - yz + c_2$$
$$-\partial v = (3xz^2 - y)\,\partial z$$
$$-v = xz^3 - yz + c_3$$

Then, adding these values of v

$$\mathbf{v = -2\,(3x^2 y + x z^3 - y z) + c}$$

where $c = c_1 + c_2 + c_3$ ☺☺

✓ **Example 1.16** *Verify whether the vector field $E = yz\,\vec{a}_x + xz\,\vec{a}_y + xy\,\vec{a}_z$ is both solenoidal and irrotational.*

☺ ***Solution:***

$$\nabla \cdot \vec{H} = \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}\right) \cdot (yz\,\vec{a}_x + xz\,\vec{a}_y + xy\,\vec{a}_z)$$

$$= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0 + 0 + 0 = 0$$

$$\nabla \cdot \vec{H} = 0$$

Hence $\vec{H}$ is solenoidal.

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \vec{a}_x\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right) - \vec{a}_y\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right) + \vec{a}_z\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)$$

$$= \vec{a}_x(x - x) - \vec{a}_y(y - y) + \vec{a}_z(z - z)$$

$$= 0$$

$$\nabla \times \vec{E} = 0$$

Hence E is irrotational. ☺☺

Example 1.17 *Determine the constant c such that the vector $\bar{F} = (x + ay)\,\bar{i} + (y + bz)\,\bar{j} + (x + cz)\,\bar{k}$ will be solenoidal.*

☺***Solution:*** If F vector is a solenoidal, then

$$\nabla \cdot \mathrm{F} = 0$$

$$\nabla \cdot \mathrm{F} = \frac{\partial}{\partial x}(x + ay) + \frac{\partial}{\partial y}(y + bz) + \frac{\partial}{\partial z}(x + cz)$$

$$= (1 + 0) + (1 + 0) + (0 + c) = 0$$

$$c = -2$$

☺☺

Example 1.18 *Show that $\bar{H} = x^2\,\bar{a}_x + y^2\,\bar{a}_y + z^2\,\bar{a}_z$ is a conservative vector field.*

☺***Solution:*** $\bar{\mathrm{H}}$ is conservative field if curl $\bar{\mathrm{H}} = 0$.

$$\text{Curl. H} = \nabla \times \mathrm{H} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \bar{a}_y\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right]$$
$$+ \bar{a}_z\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \bar{a}_x \cdot 0 + \bar{a}_y \cdot 0 + \bar{a}_z \cdot 0$$

$$= 0$$

∴ $\bar{\mathrm{H}}$ is a conservative vector field. ☺☺

Example 1.19 *Prove that $\nabla \cdot \nabla \times H = 0$.*

☺***Solution:*** Let the vector $\bar{\mathrm{H}} = \bar{a}_x \mathrm{H}_x + \bar{a}_y \mathrm{H}_y + \bar{a}_z \mathrm{H}_z$

$$\nabla \times H = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \bar{a}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \bar{a}_y\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \bar{a}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

Take Div

$$\nabla \cdot \nabla \times H = \frac{\partial}{\partial x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$= \frac{\partial^2 H_z}{\partial x\, \partial y} - \frac{\partial^2 H_y}{\partial x\, \partial z} - \frac{\partial^2 H_z}{\partial y\, \partial x} + \frac{\partial^2 H_x}{\partial y\, \partial z} + \frac{\partial^2 H_y}{\partial z\, \partial x} - \frac{\partial^2 H_x}{\partial z\, \partial y}$$

$$= 0$$

Hence, $\nabla \cdot \nabla \times H = 0$ ☺☺

Example 1.20 *Prove that* $\nabla \times \nabla V = 0$.

☺Solution: Take Curl

$$\nabla \times \nabla V = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial^2 V}{\partial y\, \partial z} - \frac{\partial^2 V}{\partial y\, \partial z}\right] - \bar{a}_y\left[\frac{\partial^2 V}{\partial x\, \partial z} - \frac{\partial^2 V}{\partial x\, \partial z}\right] + \bar{a}_z\left[\frac{\partial^2 V}{\partial x\, \partial y} - \frac{\partial^2 V}{\partial x\, \partial y}\right]$$

$$= 0$$

Hence, $\nabla \times \nabla V = 0$ ☺☺

Example 1.21 *Prove the identity* $\nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H$ *where H is a vector.*

☺Solution: Let vector $\bar{H} = \bar{a}_x H_x + \bar{a}_y H_y + \bar{a}_z H_z$

Take Curl

$$\nabla \times H = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \bar{a}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \bar{a}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \bar{a}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

Take Curl once again

$$\nabla \times \nabla \times H = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} & \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} & \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial y}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\right]$$

$$+ \bar{a}_y\left[\frac{\partial}{\partial z}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right]$$

$$+ \bar{a}_z\left[\frac{\partial}{\partial x}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\right]$$

$$= \bar{a}_x\left[\frac{\partial^2 H_y}{\partial x\, \partial y} + \frac{\partial^2 H_z}{\partial x\, \partial z}\right] - \bar{a}_x\left[\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2}\right]$$

$$+ \bar{a}_y\left[\frac{\partial^2 H_z}{\partial y\, \partial z} + \frac{\partial^2 H_x}{\partial x\, \partial y}\right] - \bar{a}_y\left[\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial z^2}\right]$$

$$+ \bar{a}_z\left[\frac{\partial^2 H_x}{\partial x\, \partial z} + \frac{\partial^2 H_y}{\partial y\, \partial z}\right] - \bar{a}_z\left[\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2}\right]$$

This can be written as

$$= \bar{a}_x \left[\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_y}{\partial x\,\partial y} + \frac{\partial^2 H_z}{\partial x\,\partial z}\right] - \bar{a}_x \left[\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2}\right]$$

$$+ \bar{a}_y \left[\frac{\partial^2 H_x}{\partial x\,\partial y} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_z}{\partial y\,\partial z}\right] - \bar{a}_y \left[\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2}\right]$$

$$+ \bar{a}_z \left[\frac{\partial^2 H_x}{\partial x\,\partial z} + \frac{\partial^2 H_y}{\partial y\,\partial z} + \frac{\partial^2 H_z}{\partial z^2}\right] - \bar{a}_z \left[\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2}\right]$$

$$= \bar{a}_x \frac{\partial}{\partial x}\left[\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right] - \bar{a}_x\, \nabla^2 H_x$$

$$+ \bar{a}_y \frac{\partial}{\partial y}\left[\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right] - \bar{a}_y\, \nabla^2 H_y$$

$$+ \bar{a}_z \frac{\partial}{\partial y}\left[\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right] - \bar{a}_z\, \nabla^2 H_z$$

$$\text{where} \quad \nabla^2 H = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2}$$

$$\nabla \times \nabla \times H = \left[\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}\right]\left[\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}\right]$$

$$- [\bar{a}_x\, \nabla^2 H_x + \bar{a}_y\, \nabla^2 H_y + \bar{a}_z\, \nabla^2 H_z]$$

$$\therefore \quad \nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H$$

$$\text{Curl Curl H} = \text{grad div H} - \nabla^2 H$$

☺☺

Example 1.22 Determine the divergence and curl of the vector

$$\vec{A} = x^2\, \vec{a}_x + y^2\, \vec{a}_y + y^2\, \vec{a}_z$$

☺Solution:

$$\vec{A} = x^2\, \vec{a}_x + y^2\, \vec{a}_y + y^2\, \vec{a}_z$$

$$\text{Div } \vec{A} = \nabla \cdot A = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(y^2)$$

$$= 2x + 2y + 0$$

$$\nabla \cdot \mathbf{A} = \mathbf{2x + 2y}$$

$$\text{Curl } \vec{A} = \nabla \times A = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & y^2 \end{vmatrix}$$

$$= \vec{a}_x\left(\frac{\partial}{\partial y} y^2 - \frac{\partial}{\partial z} y^2\right) - \vec{a}_y\left(\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial z} x^2\right) + \vec{a}_z\left(\frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} x^2\right)$$

$$\nabla \times \mathbf{A} = \mathbf{2y\, \vec{a}_x}$$

☺☺

Example 1.23 *Determine the gradient of the scalar field at $P(\sqrt{2}, \pi/2, 5)$ defined in cylindrical co-ordinate system as $A = 25\, r \sin\phi$.*

☺Solution:

$$A = 25\, r \sin\phi \text{ at } P(\sqrt{2}, \pi/2, 5)$$

$$\text{Grad. A} = \nabla A$$

$$= \left(\vec{a}_x \frac{\partial}{\partial r} + \vec{a}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \vec{a}_z \frac{\partial}{\partial z}\right)(25\, r \sin\phi)$$

$$= 25 \sin\phi\, \vec{a}_x + 25 \cos\phi\, \vec{a}_\phi$$

At $(\sqrt{2}, \pi/2, 5)$,

$$\nabla A = 25 \sin\frac{\pi}{2}\, \vec{a}_x + 25 \cos\frac{\pi}{2}\, \vec{a}_\phi$$

$$= \mathbf{25\, \vec{a}_x}$$

☺☺

Example 1.24 *Find the gradient of a scalar function of position f, where $(x, y, z) = x^2 y + e^z$. What is the magnitude and direction of the gradient at the oint $P(1, 5, -2)$?*

☺Solution:

$$f(x, y, z) = x^2 y + e^z$$

$$\text{Grad} f = \nabla f$$

$$= \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z} \right) (x^2 y + e^z)$$

$$= 2xy\, \vec{a}_x + x^2 \vec{a}_y + e^z \vec{a}_z$$

$$\nabla f(1, 5, -2) = 2 \times 5\, \vec{a}_x + \vec{a}_y + e^{-2} \vec{a}_z$$

$$= \mathbf{10}\, \vec{a}_x + \vec{a}_y + e^{-2} \vec{a}_z$$

$$\text{Magnitude} = \sqrt{10^2 + 1 + (e^{-2})^2} = 10.05$$

$$\theta = \tan^{-1}(y/x)$$

$$= \mathbf{5.7^\circ}$$

☺☺

Example 1.25 *A vectorial field is known to be defined in a rectangular co-ordinate system as,*

$$\vec{A}(x, y, z) = 5x^2 \sin \frac{\pi x}{2}\, \bar{a}_x$$

Determine the divergence of the field at a point (1, 8, 2/3) in the three dimensional space.

☺**Solution:** Given the vector field,

$$\vec{A} = 5x^2 \sin \frac{\pi x}{2}\, \bar{a}_x$$

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = 10x \sin \frac{\pi x}{2} + 5x^2 \cdot \pi \cos \frac{\pi x}{2}$$

$$\text{div } \vec{A}\,(x = 1) = 10 \sin \frac{\pi}{2} = \mathbf{10}$$

☺☺

Example 1.26 *Determine the expressions for divergence, gradient and cur in cylindrical and spherical coordinate systems.*

☺ **Solution:** Cylindrical coordinate system:

$$\text{div. } \bar{A} = \nabla \cdot \bar{A}$$

$$= \frac{1}{r} \frac{\partial}{\partial r} (r \bar{A}_r) + \frac{1}{r} \frac{\partial \bar{A}_\phi}{\partial \phi} + \frac{\partial \bar{A}_z}{\partial z}$$

$$\text{gra. } v = \nabla v$$

$$= \frac{\partial v}{\partial r} \bar{a}_r + \frac{1}{r} \frac{\partial v}{\partial \phi} \bar{a}_\phi + \frac{\partial v}{\partial z} \bar{a}_z$$

$$\text{curl } \bar{A} = \nabla \times A$$

$$= \frac{1}{r} \left[\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right] \bar{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \right] \bar{a}_\phi$$

$$+ \left[\frac{1}{r} \frac{\partial (r\, H_\phi)}{\partial r} - \frac{1}{r} \frac{\partial H_r}{\partial \phi} \right] \bar{a}_z$$

Spherical coordinate system

$$\text{div. } \bar{A} = \nabla \cdot \bar{A}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \bar{A}_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta\, D_\theta) + \frac{1}{r \sin\theta} \frac{\partial D_\phi}{\partial \phi}$$

$$\text{grad. } v = \nabla v$$

$$= \frac{\partial v}{\partial r} \bar{a}_r + \frac{1}{r} \frac{\partial v}{\partial \theta} \bar{a}_\theta + \frac{1}{r \sin\theta} \frac{\partial v}{\partial \phi} \bar{a}_\phi$$

$$\text{curl } \bar{A} = \nabla \times \bar{A}$$

$$= \frac{1}{r \sin\theta} \left[\frac{\partial H_\phi}{\partial \theta} \sin\theta - \frac{\partial H_\theta}{\partial \phi} \right] \bar{a}_r +$$

$$\frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial (r\, H_\phi)}{\partial r} \right] \bar{a}_\phi + \frac{1}{r} \left[\frac{\partial (r\, H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right] \bar{a}_\phi$$

☺☺

Example 1.27 *Transform the Cartesian co-ordinates $x = 2, y = 1, z = 3$ into spherical co-ordinates.*

☺ ***Solution:***

Given	*Transform*
$x = 2$	$r = \sqrt{x^2 + y^2 + z^2}$ $= \sqrt{4+1+9}$ $= \sqrt{14} = 3.74$
$y = 1$	$\theta = \cos^{-1}\left(\frac{z}{r}\right)$ $= \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.7°$
$z = 3$	$\phi = \tan^{-1}(y/x)$ $= \tan^{-1}\left(\frac{1}{2}\right) = 26.56°$

Spherical co-ordinates are (3.74, 36.7°, 26.56°). ☺☺

Example 1.28 *Give the cartesian co-ordinates of a point whose cylindrical co-ordinates are $\rho = 1$, $\phi = 45°$, $z = 2$.*

☺***Solution:*** Cylindrical co-ordinates are $\rho = 1$, $\phi = 45°$, $z = 2$.

Cartesian co-ordinates are x, y, z.

$$x = \rho \cos \phi$$
$$= 1 \cdot \cos 45° = 0.707$$
$$y = \rho \sin \phi$$
$$= 1 \cdot \sin 45° = 0.707$$
$$z = z$$
$$= 2$$

Cartesian co-ordinates are (0.707, 0.707, 2) ☺☺

Example 1.29 *Transform the cartesian co-ordinates (2, 1, 3) into spherical co-ordinates.*

☺***Solution:***

Given: Cartesian co-ordinates $x = 2,\ y = 1,\ z = 3$

Cylindrical co-ordinates

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+1} = \sqrt{5} = 2.236$$
$$\phi = \tan^{-1}(y/x) = \tan^{-1}\frac{1}{2} = 26.56°$$
$$z = 3$$
$$(\rho, \phi, z) = \mathbf{(2.236,\ 26.56°,\ 3)}$$

☺☺

Example 1.30 *Express this vector $A = 4\,\bar{a}_x + 2\,\bar{a}_y + 3\,\bar{a}_z$ in cylindrical co-ordinate system.*

☺***Solution:***

$$A = 4\,\bar{a}_x + 2\,\bar{a}_y + 3\,\bar{a}_z$$
$$A_x = 4,\ A_y = 2,\ A_z = 3$$
$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{4} = 26.56°$$
$$A_\rho = A_x \cos\phi + A_y \sin\phi$$
$$= 4\cos 26.56° + 2\sin 26.56°$$
$$= 4.46$$
$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$
$$= -4\sin 26.56° + 2\cos 26.56°$$
$$= 0$$
$$A_z = 3$$

In cylindrical co-ordinates, $A = A_\rho\,\bar{a}_\rho + A_\phi\,\bar{a}_\phi + A_z\,\bar{a}_z$

$$= \mathbf{4.46\,\bar{a}_\rho + 3\,\bar{a}_z}$$

☺☺

Example 1.31 *Given the two points.*

$$A(x = 2, y = 3, z = -1) \quad \text{and} \quad B(r = 4, \theta = 25°, \phi = 120°).$$

Find the spherical co-ordinates of A and cartesian co-ordinates of B.

☺***Solution:*** Given cartesian co-ordinates $A(x = 2,\ y = 3,\ z = -1)$.

$$r^2 = x^2 + y^2 + z^2 = 2^2 + 3^2 + (-1)^2 = 14$$

$$r = \sqrt{14}$$

$$\cos\theta = \frac{z}{r} = \frac{-1}{\sqrt{14}}$$

$$\theta = 105.5°$$

$$\tan\phi = \frac{y}{x} = \frac{3}{2}$$

$$\phi = 56.3°$$

Spherical co-ordinates of A ($r = \sqrt{14}$, $\theta = 105.5°$, $\phi = 56.3°$).

Given spherical co-ordinates B ($r = 4, \theta = 25°, \phi = 120°$)

$$x = r \sin\theta \cos\phi$$

$$= 4 \sin 25° \cos 120° = -0.845$$

$$y = r \sin\theta \sin\phi$$

$$= 4 \sin 25° \sin 120° = 1.385$$

$$z = r \cos\theta$$

$$= 4 \cos 25° = 3.625$$

Cartesian co-ordinates of B ($x = -0.845, y = 1.385, z = 3.625$)

Example 1.32 *A vector field is given by the expression:*

(a) $F = \left(\frac{1}{r}\right) \vec{u_r}$ *(in cylindrical co-ordinates)*

(b) $F = \left(\frac{1}{R}\right) \vec{u_R}$ *(in spherical co-ordinates)*

Determine $\vec{F}$ is each case in cartesian form, at a point with $x = 1, y = 1$, and $z = 1$ unit.

☺***Solution:***

(a)

$$F = \left(\frac{1}{r}\right) u_r \quad \text{in cylindrical form}$$

$$= F_r \vec{u_r} + F_\phi \vec{u_r} + F_z \vec{u_r} \quad \text{is general form.}$$

$$\therefore F_\phi = 0 \quad \text{and} \quad F_z = 0$$

$$F_r = \frac{1}{r}$$

At point ($x = 1,\ y = 1,\ z = 1$)

$$r = \sqrt{x^2 + y^2} = \sqrt{1 + 1} \qquad [\because z = 0]$$

$$= \sqrt{2}$$

$$F_r = \frac{1}{r} = \frac{1}{\sqrt{2}}$$

$$\text{But } F_x = F_r \cdot \frac{x}{\sqrt{x^2 + y^2}} \qquad [\because F_\phi = 0]$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$F_y = F_r \frac{y}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

The field at (1, 1, 1) in a cartesian co-ordinate

$$F = F_x \vec{u_x} + F_y \vec{u_y} + F_z \vec{u_z} = \frac{1}{2} \vec{u_x} + \frac{1}{2} \vec{u_y}$$

(b)

$$F = \left(\frac{1}{R}\right) u_R \text{ in spherical form.}$$

$$= F_R \vec{u_R} + F_\theta \vec{u_\theta} + F_\phi \vec{u_\phi} \quad \text{in general form.}$$

$$\therefore F_\theta = 0 \quad \text{and} \quad F_\phi = 0$$

$$F_R = \frac{1}{R}$$

At point ($x = 1,\ y = 1,\ z = 1$)

$$R = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$F_R = \frac{1}{R} = \frac{1}{\sqrt{3}}$$

$$F_x = F_R\left(\frac{x}{R}\right) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

[$\because F_\theta = 0\ \&\ F_\phi = 0$]

$$F_y = F_R\left(\frac{y}{R}\right) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

$$F_z = F_R\left(\frac{z}{R}\right) = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$$

The field at (1, 1, 1) in a cartesian co-ordinate

$$F = F_x \vec{u_x} + F_y \vec{u_y} + F_z \vec{u_z}$$

$$= \frac{1}{3} \vec{u_x} + \frac{1}{3} \vec{u_y} + \frac{1}{3} \vec{u_z}$$

☺☺

Example 1.33 *Transform the vector $\bar{A} = 4\,\bar{a}_x - 2\,\bar{a}_y - 4\,\bar{a}_z$ at p(x = 2, y = 3, z = 4) to spherical coordinate.*

☺Solution: $A = 4\bar{a}_x - 2\bar{a}_y - 4\bar{a}_z$

$$A_x = 4, \quad A_y = -2, \quad A_z = -4$$

Spherical coordinates in terms of rectangular coordinates

$$\bar{A}_r = \bar{A}_x \bar{a}_x \cdot \bar{a}_r + \bar{A}_y \bar{a}_y \cdot \bar{a}_r + \bar{A}_z \bar{a}_z \cdot \bar{a}_r$$

$$\bar{A}_\theta = \bar{A}_x \bar{a}_x \cdot \bar{a}_\theta + \bar{A}_y \bar{a}_y \cdot \bar{a}_\theta + \bar{A}_z \bar{a}_z \cdot \bar{a}_\theta$$

$$\bar{A}_\phi = \bar{A}_x \bar{a}_x \cdot \bar{a}_\phi + \bar{A}_y \bar{a}_y \cdot \bar{a}_\phi + \bar{A}_z \bar{a}_z \cdot \bar{a}_\phi$$

But $\bar{a}_x \cdot \bar{a}_r = \sin\theta\cos\phi, \quad \bar{a}_y \cdot \bar{a}_r = \sin\theta\sin\phi$

$$\bar{a}_z \cdot \bar{a}_r = \cos\theta$$

$$\bar{a}_x \cdot \bar{a}_\theta = \cos\theta\cos\phi, \quad \bar{a}_y \cdot \bar{a}_\theta = \cos\theta\sin\phi$$

$$\bar{a}_z \cdot \bar{a}_\theta = -\sin\theta$$

$$\bar{a}_x \cdot \bar{a}_\phi = -\sin\phi, \quad \bar{a}_y \cdot \bar{a}_\phi = \cos\phi$$

$$\bar{a}_z \cdot \bar{a}_\phi = 0$$

Substituting these values in spherical coordinates

$$\bar{A}_r = 4\sin\theta\cos\phi - 2\sin\theta\sin\phi - 4\cos\theta$$

$$\bar{A}_\theta = 4\cos\theta\cos\phi - 2\cos\theta\sin\phi + 4\sin\theta$$

$$\bar{A}_\phi = -4\sin\phi - 2\cos\phi$$

At point ($x = 2, y = 3, z = 4$)

$$\phi = \tan^{-1}(y/x) = \tan^{-1}(3/2) = 56.3°$$

$$\theta = \cos^{-1}(z/r)$$

$$= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\frac{4}{\sqrt{2^2+3^2+4^2}}$$

$$= 42.03°$$

☺☺

Example 1.34 *Using Divergence theorem, evaluate $\iint_S \bar{F} \cdot n\ ds$ where $\bar{F} = 2\,xy\,\bar{i} + y^2\,\bar{j} + 4\,yz\,\bar{k}$ and S is the surface of the cube bounded by x = 0, x = 1, y = 0, y = 1; and z = 0, z = 1.*

☺Solution: By divergence theorem,

$$\iint_S \bar{F} \cdot n\,ds = \iiint_V \nabla \cdot \bar{F}\,dv$$

$$\nabla \cdot \bar{F} = \left(\bar{i}\,\frac{\partial}{\partial x} + \bar{j}\,\frac{\partial}{\partial y} + \bar{k}\,\frac{\partial}{\partial z}\right) \cdot (2\,xy\,\bar{i} + y^2\bar{j} + 4\,yz\,\bar{k})$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(4yz)$$

$$= 2y + 2y + 4y = 8y$$

$$\iiint \hat{\nabla} \cdot \bar{F}\, dv = \int_0^1\int_0^1\int_0^1 8y\, dx\, dy\, dz = \int_0^1\int_0^1 [8yz]_0^1\, dx\, dy$$

$$= \int_0^1\int_0^1 8y\, dx\, dy$$

$$= \int_0^1 \left[8\frac{y^2}{2}\right]_0^1 dx = 4\int_0^1 dx = 4[x]_0^1 = 4$$

$$\iiint \bar{F} \cdot dv = 4$$

☺☺

Example 1.35 *Check validity of the divergence theorem considering the field* $D = 2xy\,\bar{a}_x + x^2\,\bar{a}_y$ *c/m²* *and the rectangular parallelopiped formed by the planes* $x = 0, x = 1, y = 0, y = 2;$ *and* $z = 0, z = 3$.

☺***Solution:*** By divergence theorem,

$$\iint \bar{D} \cdot n\, ds = \iiint_V \nabla \cdot \bar{D}\, dv$$

$$\nabla \cdot \bar{D} = \left(\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}\right) \cdot (2xy\,\bar{a}_x + x^2\,\bar{a}_y)$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + 0$$

$$= 2y + 0 = 2y$$

$$\iiint \nabla \cdot \bar{D}\, dv = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} 2y\, dx\, dy\, dz$$

$$= \int_0^1\int_0^2 [2yz]_0^3\, dx\, dy = \int_0^1\int_0^2 6y\, dx\, dy$$

$$= \int_0^1 \left[6\frac{y^2}{2}\right]_0^2 dx = \int_0^1 12\, dx = [12x]_0^1 = 12$$

Evaluation of $\iint D \cdot n\, ds$

$$\iint D \cdot n\, ds = \iint D \cdot \bar{a}_x\, dy\, dz + \iint D \cdot (-\bar{a}_x)\, dy\, dz + \iint D \cdot \bar{a}_y\, dx\, dz$$
$$+ \iint D \cdot (-\bar{a}_y)\, dx\, dz + \iint D \cdot \bar{a}_z\, dx\, dy + \iint D \cdot (-\bar{a}_z)\, dx\, dy$$

$$\iint D \cdot \bar{a}_x\, dy\, dz = \iint (2xy\,\bar{a}_x + x^2\,\bar{a}_y)\,(\bar{a}_x)\, dy\, dz$$

$$= \int_0^2\int_0^3 2xy\, dy\, dz$$

$$= \int_0^2 [2xyz]_0^3\, dy = \int_0^2 6xy\, dy$$

$$= \left[6x\frac{y^2}{2}\right]_0^2 = 12x = 12 \qquad [\because x = 1]$$

$$\iint D \cdot (-\bar{a}_x)\, dy\, dz = \iint (2xy\,\bar{a}_x + x^2\,\bar{a}_y)\,(-\bar{a}_x)\, dy\, dz$$

$$= 0 \qquad [\because x = 0]$$

$$\iint D \cdot \bar{a}_y\, dx\, dz = \iint (2xy\,\bar{a}_x + x^2\,\bar{a}_y)\,(\bar{a}_y)\, dx\, dz$$

$$= \int_0^1\int_0^3 x^2\, dx\, dz = \int_0^1 [x^2 z]_0^3\, dx$$

$$= \int_0^1 3x^2\, dx = 3\left.\frac{x^3}{3}\right|_0^1 = 1$$

$$\iint D\cdot(-\bar{a}_y)\,dx\,dz = \iint (2xy\,\bar{a}_x + x^2\,\bar{a}_y)(-\bar{a}_y)\,dx\,dz$$

$$= -\int_0^1\int_0^3 x^2\,dx\,dz = -\int_0^1 [x^2 z]_0^3\,dx$$

$$= -3\int_0^1 x^2\,dx = -3\left.\frac{x^3}{3}\right|_0^1 = 1$$

$$\iint D\cdot\bar{a}_z\,dx\,dy = \iint D\cdot(-\bar{a}_z)\,dx\,dy \qquad [\because \bar{a}_z = 0$$

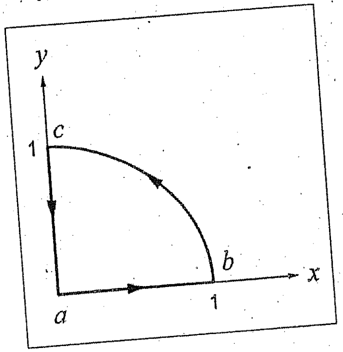
$$= 0$$

$$\therefore \iint D\cdot n\,ds = 12+0+1-1 = 12$$

Hence, $\iint D\cdot n\,ds = \iiint_v \nabla\cdot D\,dv$

☺

Example 1.36 *Given $\bar{A} = 2r\cos\phi\,\hat{I}_r + r\,\hat{I}_\phi$ in cylindrical co-ordinates. Fo the contour shown in Fig., verify Stoke's theorem.*



☺Solution: According to Stoke's theorem,

$$\oint_c A\cdot dl = \iint_S \nabla\times A\;ds$$

$$A = 2r\cos\phi\,\hat{I}_r + r\,\hat{I}_\phi$$

where $\hat{I}_r$ and $\hat{I}_\phi$ are unit vectors.

Converting cylindrical co-ordinates to cartesian co-ordinates.

$$A_x = \frac{x}{\sqrt{x^2+y^2}}A_r + \frac{y}{\sqrt{x^2+y^2}}A_\phi$$

$$= \frac{x}{\sqrt{x^2+y^2}}\,2r\cos\phi + \frac{y}{\sqrt{x^2+y^2}}\,r$$

$$A_x = 2x\cos\phi + y \qquad [\because r = \sqrt{x^2+y^2}]$$

Similarly,

$$A_y = \frac{y}{\sqrt{x^2+y^2}}A_r + \frac{x}{\sqrt{x^2+y^2}}A_\phi$$

$$= \frac{y}{\sqrt{x^2+y^2}}\,2r\cos\phi + \frac{y}{\sqrt{x^2+y^2}}\,r$$

$$A_y = 2y\cos\phi + x \qquad [\because r = \sqrt{x^2+y^2}]$$

$$A = A_x\,\hat{I}_x + A_y\,\hat{I}_y$$

$$A = (2x\cos\phi + y)\,\hat{I}_x + (2y\cos\phi + x)\,\hat{I}_y$$

$$\nabla\times A = \begin{vmatrix} \hat{I}_x & \hat{I}_y & \hat{I}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x\cos\phi + y & 2y\cos\phi + x & 0 \end{vmatrix}$$

$$= \hat{I}_x(0-0) - \hat{I}_y(0-0) + \hat{I}_z(1-1) = 0$$

$$\iint \nabla\times A = 0$$

$$\oint A\,dl = \int_a^b A_{ab}\cdot dl + \int_b^c A_{bc}\cdot dl + \int_c^a A_{ca}\cdot dl$$

where A_{ab}, A_{bc}, A_{ca} are the expressions of A evaluated for segments *ab*, *bc* and *ca* respectively.

$$\oint A \cdot dl = \int_0^1 (2x\cos\phi + y)\hat{I}_x dx + \int_1^1 [(2x\cos\phi + y)\hat{I}_x + (2y\cos\phi + x)\hat{I}_y]$$

$$\cdot [\hat{I}_x dx + \hat{I}_y dy] + \int_1^0 (2y\cos\phi + x)\hat{I}_y dy$$

$$= [x^2\cos\phi]_0^1 + [x^2\cos\phi + y^2\cos\phi]_1^1 + [y^2\cos\phi]_1^0$$

$$= \cos\phi + 0 - \cos\phi = 0$$

$$\therefore \oint A \cdot dl = 0$$

$$\iint \nabla \times A = \int A \cdot dl.$$ Hence proved. ☺☺

Example 1.37 *A vector field* $D = \left[\frac{5r^2}{4}\right]\bar{I}_r$ *is given in spherical co-ordinates. Evaluate both sides of divergence theorem for the volume enclosed between r = 1 and r = 2.*

☺***Solution:*** By divergence theorem,

$$\iiint \nabla \cdot D \, dv = \oint\!\!\oint_s D \cdot n \, ds$$

$$D = \left[\frac{5r^2}{4}\right]\bar{I}_r$$

$$\nabla \cdot D = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}(r^2 \sin\theta \, D_r)\right) \qquad [\because D_\theta = D_\phi = 0]$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\left(\frac{5r^2}{4}\right)\right)\right)$$

$$= \frac{1}{r^2\sin\theta}\left[\left(\frac{5}{4}\sin\theta\right)\frac{\partial}{\partial r}(r^4)\right]$$

$$= \frac{5}{4r^2}\left[\frac{\partial}{\partial r}(r^4)\right] = \frac{5}{4r^2}\cdot 4r^3 = 5r$$

LHS of divergence theorem,

$$\iiint_V \nabla \cdot D \, dv = \int_1^2 \int_0^\pi \int_0^{2\pi} 5r \cdot r^2 \sin\theta \cdot d\theta \cdot d\phi \cdot dr$$

$$= \int_1^2 \int_0^\pi 5r^3 \sin\theta \cdot d\theta \cdot dr \, (2\pi - 0)$$

$$= 10\pi \int_1^2 \int_0^\pi \sin\theta \cdot d\theta \cdot r^3 \, dr$$

$$= 10\pi \int_1^2 (-\cos\theta)_0^\pi \, r^3 \, dr$$

$$= 10\pi \int_1^2 [-(-1-1)] \, r^3 \, dr$$

$$= 10\pi \int_1^2 2r^3 \, dr = 20\pi \left[\frac{r^4}{4}\right]_1^2$$

$$= 20\pi\left[\frac{16}{4} - \frac{1}{4}\right]$$

$$\iiint \nabla \cdot \mathbf{D} \, dv = 75\pi$$

RHS of divergence theorem.

$$\oint\!\!\oint_s D \cdot \bar{n} \, ds = 8 \int_1^2 \int_0^{2\pi} \frac{5r^2}{4} r \cdot dr \cdot d\theta$$

$$= 8 \int_1^2 \int_0^{2\pi} d\theta \left[\frac{5r^2}{4}\right] r \cdot dr$$

$$= 8 \int_1^2 2\pi \frac{5}{4} r^3 \, dr = \frac{80\pi}{4} \int_1^2 r^3 \, dr$$

$$= \frac{80\pi}{4}\left[\frac{r^4}{4}\right]_1^2$$

$$= \frac{80\pi}{4}\left[\frac{16}{4} - \frac{1}{4}\right]$$

$$= 75\pi$$

$$\iiint_V \nabla \cdot \mathbf{D}\, dv = \oint_s \mathbf{D} \cdot \bar{n}\, ds = 75\pi$$

☺☺

Example 1.38 *Prove that divergence of a curl of a vector is zero, using Stoke's theorem.*

☺**Solution:** Consider vector A.

$$\text{Curl of A} = \nabla \times A \text{ is a vector}$$

$$\text{Divergence of } \nabla \times A = \nabla \cdot (\nabla \times A) \text{ is a scalar say } \alpha.$$

$$\therefore \quad \nabla \cdot \nabla \times A = \alpha$$

Integrating both sides over a volume, we get

$$\int_{vol} \nabla \cdot (\nabla \times A)\; dv = \int_{vol} \alpha \cdot dv$$

By applying Divergence theorem, the left hand side can be converted to a surface integral.

$$\therefore \int_{vol} \nabla \cdot (\nabla \times A)\; dv = \int_s (\nabla \times A) \cdot ds$$

$$= \int_{vol} \alpha \cdot dv$$

The Stoke's theorem applied to closed surface produces zero. $[\therefore \alpha \cdot dv = 0]$

$$\int_{vol} \alpha \cdot dv = 0$$

$$\text{But} \quad dv \neq 0$$

$$\therefore \quad \alpha = 0$$

$$\nabla \cdot (\nabla \times A) = 0$$

☺☺

Example 1.39 *State and verify Divergence theorem for the vector* $A = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ *taken over the cube bounded by* $x = 0$, $x = 1$, $y = 0$, $y = 1$.

☺**Solution:**

Divergence theorem: The volume integral of the divergence of a vector field over a volume is equal to the surface integral of the normal component of this vector over the surface bounding the volume.

$$\iiint_V \nabla \cdot A\, dv = \oint_S A \cdot ds$$

Given:

$$A = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$$

$$\nabla \cdot A = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})$$

$$= 4 - 4y + 2z$$

$$\iiint \nabla \cdot A = \int_0^1\int_0^1\int_0^1 (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_0^1\int_0^1 4z - 4yz + \frac{2}{2}z^2 \Big|_0^1 dx\, dy$$

$$= \int_0^1\int_0^1 (5 - 4y)\, dx\, dy$$

$$= \int_0^1 (5y - \frac{4}{2}y^2)\Big|_0^1 dx$$

$$= \int_0^1 3\, dx$$

$$= 3x \Big|_0^1 = 3$$

$$\oint A\cdot ds = \iint A\cdot \vec{i}\; dy\,dz + \iint A(-\vec{i}\,)\,dy\,dz + \iint A\,\vec{j}\;\, dx\,dz$$

$$+ \iint A(-\vec{j}\,)\,dx\,dz + \iint A\,\vec{k}\;\, dx\,dy + \iint A\,(-\vec{k}\,)\,dx\,dy$$

$$\iint A\cdot \vec{i}\; x\,dy\,dz = \int_0^1\!\!\int_0^1 (4x\,\vec{i} - 2y^2\vec{j} + z^2\vec{k}\,)\,\vec{i}\;\, dy\,dz$$

$$= \int_0^1\!\!\int_0^1 4x\,dy\,dz$$

$$= \int_0^1 4xy\,\Big|_0^1\, dz$$

$$= 4xz\,\Big|_0^1 = 4x$$

$$= 4 \qquad [\because x = 1]$$

$$\iint A(-\vec{i}\,)\,dy\,dz = 0 \qquad [\because x = 0]$$

$$\iint A(\vec{j}\,)\,dx\,dz = \int_0^1\!\!\int_0^1 (4x\,\vec{i} - 2y^2\vec{j} + z^2\vec{k}\,)\;\vec{j}\,dy\,dz$$

$$= \int_0^1\!\!\int_0^1 -2y^2\,dx\,dz$$

$$= \int_0^1 -2xy^2\,\Big|_0^1\, dz$$

$$= -2y^2 z\,\Big|_0^1$$

$$= -2y^2$$

$$= -2 \qquad [\because y = 1]$$

$$\iint A(-\vec{j}\,)\,dx\,dz = 0 \qquad [\because y = 0]$$

$$\iint A\,\vec{k}\;\, dx\,dy = \int_0^1\!\!\int_0^1 (4x\,\vec{i} - 2y^2\vec{j} + z^2\vec{k}\,)\;\vec{k}\;\, dx\,dy$$

$$= \int_0^1\!\!\int_0^1 z^2\,dx\,dy$$

$$= \int_0^1 z^2 x\,dy\,\Big|_0^1$$

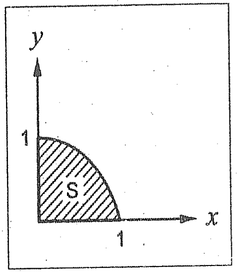
$$= z^2 y\,\Big|_0^1 = z^2 = 1 \qquad [\because z = 1]$$

$$\iint A\cdot(-\vec{k}\,)\,dx\,dy = 0 \qquad [\because z = 0]$$

$$\oint A\cdot ds = 4 - 2 + 1 = 3$$

Hence it is verified. ☺☺

Example 1.40 *Given* $A = \rho\,\cos\,\theta\;\vec{a}_\rho + \rho^2\;\vec{a}_z$. *Compute* $\nabla\times A$ *and* $\int_S \nabla\times A\cdot ds$ *over the area S as shown in Figure.*



☺ **Solution:**

$$\nabla\times A = \frac{1}{\rho}\begin{vmatrix} \vec{a}_\rho & \rho\,\vec{a}_\phi & \vec{a}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho\,A_\phi & A_z \end{vmatrix}$$

$$A_\rho = \rho\cos\theta,\quad A_\phi = 0,\quad A_z = \rho^2$$

$$\nabla\times A = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial z}\right]\vec{a}_\rho + 1\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\vec{a}_\phi + \left[\frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right]\vec{a}_z$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2)\,\bar{a}_\rho + \rho\frac{\partial}{\partial z}(\rho\cos\phi)\,\bar{a}_\phi - \frac{\partial(\rho^2)}{\partial\rho}\,\bar{a}_\phi - \frac{1}{\rho}\frac{\partial(\rho\cos\phi)}{\partial\phi}\,\bar{a}_z$$

$$= -2\rho\,\vec{a}_\phi + \sin\phi\,\vec{a}_z.$$

$$\int\nabla\times A\cdot ds = \int_S(-2\rho\,\vec{a}_\phi + \sin\phi\,\vec{a}_z)\,\rho\,d\rho\,d\phi\,\vec{a}_z$$

where $ds = \rho\,d\rho\,d\phi\,\vec{a}_z$

$$\int\nabla\times A\,ds = \int_{\phi=0}^{90^\circ}\int_{\rho=0}^{1}(-2\rho^2\,d\rho\,d\phi\,\vec{a}_\phi\,\vec{a}_z + \rho\sin\phi\,\partial\rho\,d\phi)$$

$$= \int_{\phi=0}^{90^\circ}\frac{\rho^2}{2}\Big|_0^1\sin\phi\,d\phi = \frac{1}{2}(-\cos\phi)_0^{90} = \frac{1}{2}$$

☺☺

Example 1.41 *In region $r \le a$ in spherical co-ordinates, $E = \frac{r\rho}{3\varepsilon}\bar{r}$ show tha $\oint E\cdot ds = \int(\nabla\cdot E)\,dv$ by divergence theorem.*

☺Solution:

$$E = \frac{r\rho}{3\varepsilon}\bar{r}$$

$$\oint E\cdot ds = \oint\frac{r\rho}{3\varepsilon}\bar{r}\,ds \qquad [\because ds = \int 4\pi r^2 \text{ surface area}$$

For spherical co-ordinates

$$\nabla\cdot E = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\,E_r) + \frac{\partial}{\partial\theta}(r\sin\theta\,E_\theta) + \frac{\partial}{\partial\phi}(r\,E_\phi)\right]$$

Since E_θ and E_ϕ are zero, then

$$\nabla\cdot E = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\,E_r)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta\,r\,\rho}{3\varepsilon}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{3r^2\rho\sin\theta}{3\varepsilon}\right] = \frac{\rho}{\varepsilon}$$

$$\int_V\nabla\cdot E\,dv = \int\frac{\rho}{\varepsilon}\cdot dv = \frac{\rho}{\varepsilon}\frac{4\pi r^3}{3} \qquad \left[\because \int dv = \frac{4\pi r^3}{3},\ \text{volume}\right]$$

Hence $\int E\cdot ds = \int(\nabla\cdot E)\,dv$ ☺☺

Example 1.42 *A charge $Q_2 = 121\times10^{-9}$ C is located in vacuum at $P_2(-0.03,\ 0.01,\ -0.04)$. Find the force on Q_2 due to $Q_1 = 100\ \mu C$ at $P_1(0.03,\ 0.08,\ 0.02)$. All distances are in meters.*

☺Solution: $Q_1 = 100\ \mu C$ $\quad P_1(0.03, 0.08, 0.02)$

$Q_2 = 121$ nC $\quad P_2(-0.03, 0.01, -0.04)$

The distance between any two points say (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$$

$$= \sqrt{(0.03+0.03)^2 + (0.08-0.01)^2 + (0.02+0.04)^2}$$

$$= \sqrt{0.0121}$$

$$d = 0.11\text{ m}$$

According to Coulomb's law, force between any two charge Q_1 and Q_2 is given by

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 d^2}$$

$$= \frac{100 \times 10^{-6} \times 121 \times 10^{-9}}{4 \times 3.14 \times 8.854 \times 10^{-12} \times (0.11)^2} \text{ Newtons}$$

$$F = 8.989 \text{ Newtons}$$

☺☺

∴ The force on Q_2 due to Q_1 is 8.989 Newtons.

Example 1.43 *Compute the force between two charges of 1 C, eac separated in vacuum by a distance equal to (i) 1 metre, (ii) the diameter of th earth (1.27×10^7 m).*

☺Solution:

(i) $Q_1 = Q_2 = 1\text{ C}$

$r = 1\text{ m}$

$$\text{Force } F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} = \frac{1}{4\pi \times \frac{10^{-9}}{36\pi} \times 1} = \mathbf{9 \times 10^9 \text{ Newtons}}$$

(ii) $r = 1.27 \times 10^7\text{ m}$

$$F = \frac{1}{4\pi \times \frac{10^{-9}}{36\pi} \times (1.27 \times 10^7)^2} = \mathbf{7.087 \times 10^{-5} \text{ Newtons}}$$

☺☺

Example 1.44 *A point charge of 10 μc is located at (1, 2, 3) and anoth point charge of – 3 μc is located at (3, 0, 2) in vacuum. Find the force betwe them.*

☺Solution: $Q_1 = 10\ \mu c = 10 \times 10^{-6}\text{ C}$ $\qquad r_1 = \bar{a}_x + 2\bar{a}_y + 3\bar{a}_z$

$Q_2 = -2\ \mu c = -3 \times 10^{-6}\text{ C}$ $\qquad r_2 = 3\bar{a}_x + 2\bar{a}_z$

$$r = r_2 - r_1 = (3-1)\bar{a}_x + (0-2)\bar{a}_y + (2-3)\bar{a}_z$$

$$= 2\bar{a}_x - 2\bar{a}_y - \bar{a}_z$$

$$|r| = \sqrt{4+4+1} = 3$$

$$\text{Unit vector } \bar{a} = \frac{r}{|r|} = \frac{2\bar{a}_x - 2\bar{a}_y - \bar{a}_z}{3}$$

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2}\bar{a}$$

$$= \frac{10 \times 10^{-6} \times (-3) \times 10^{-6}}{4\pi \times \frac{1}{36\pi \times 10^9} \times (3)^2}\left(\frac{2\bar{a}_x - 2\bar{a}_y - \bar{a}_z}{3}\right)$$

$$= \mathbf{(2\bar{a}_x - 2\bar{a}_y - \bar{a}_z) \times 10^{-2} \text{ Newtons}}$$

☺☺

Example 1.45 *Two small identical conducting spheres have charge of –1 nano coloumbs and 2 nano coulombs respectively. If they are brought in contact and then separated by 4 cm. What is the force between them?*

☺Solution: $Q_1 = -1 \times 10^{-9}\text{ C}$ $\qquad r = 4 \times 10^{-2}\text{ m}$

$Q_2 = 2 \times 10^{-9}\text{ C}$

When two charges Q_1 and Q_2 are brought in contact and separated, the new charge is given by

$$Q = \frac{Q_1 + Q_2}{2} = \frac{(-1+2) \times 10^{-9}}{2} = 0.5 \times 10^{-9}\text{ C}$$

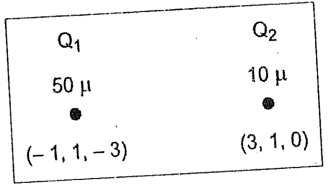
$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} \qquad [\because Q_1 = Q_2 = Q]$$

$$= \frac{(0.5 \times 10^{-9})^2}{4\pi \times \frac{1}{36\pi \times 10^9} \times 4 \times 10^{-4}}$$

$$= \frac{0.25 \times 9 \times 10^{-5}}{4} = \mathbf{5.625 \text{ Micro Newtons}}$$

☺☺

Example 1.46 *If two charges of 50 μc and 10 μc are located at (–1, 1, 3) and (3, 1, 0) respectively. Determine the nature of force and its magnitude and direction, existing between the two point charges.*



☺***Solution:***

$$Q_1 = 50 \times 10^{-6} \text{ c at } r_1 = -\bar{a}_x + \bar{a}_y + 3\,\bar{a}_z$$

$$Q_2 = 10 \times 10^{-6} \text{ c at } r_2 = 3\,\bar{a}_x + \bar{a}_y$$

$$r = r_2 - r_1 = 4\,\bar{a}_x - 3\,\bar{a}_z$$

$$\bar{a} = \frac{r}{|r|} = \frac{4\,\bar{a}_x - 3\,\bar{a}_z}{\sqrt{4^2+3^2}}$$

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\,\bar{a}$$

(*i*) $$\vec{F} = \frac{50\,\mu \times 10\,\mu\,(4\,\bar{a}_x + 0\,\bar{a}_y - 3\,\bar{a}_z)}{4\pi\varepsilon_0\,[4^2+3^2]^{3/2}} \quad \text{from coulomb's law}$$

$$\vec{F} = 0.036\,(4\,\bar{a}_x + 0\,\bar{a}_y - 3\,\bar{a}_z)$$

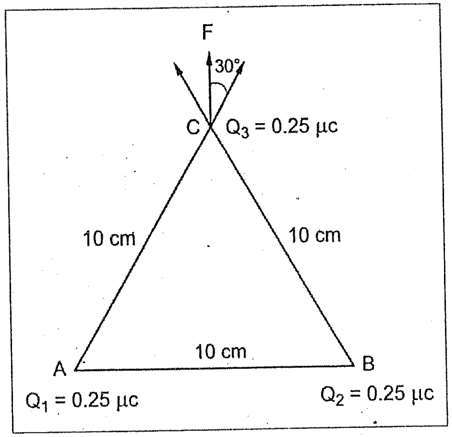
$$\vec{F} = \mathbf{(0.144\,\bar{a}_x - 0.108\,\bar{a}_z)\ \ Newton}$$

☺

Example 1.47 *Three concentrated charge of 0.25 μc are located at th vertices of a equilateral triangle of 10 cm side. Find the magnitude and direction the force on one charge due to the other two charges.*

☺***Solution:*** The force on Q_3 due to charge Q_1

$$|F_{13}| = \frac{Q_1 Q_3}{4\pi\varepsilon_0 r^2} = \frac{(0.25)^2 \times 10^{-6} \times 10^{-6}}{4\pi \times \dfrac{1}{36\pi \times 10^9} \times (0.1)^2} = 0.05625 \text{ Newtons}$$



The force on Q_3 due to charge Q_2.

$$|F_{23}| = \frac{Q_2 Q_3}{4\pi\varepsilon_0 r^2} = \frac{(0.25)^2 \times 10^{-12}}{4\pi \times \dfrac{1}{36\pi \times 10^9} \times (0.1)^2}$$

$$= 0.05625 \text{ Newtons}$$

The resultant force, $F = |F_{13}|\cos 30° + |F_{23}|\cos 30°$

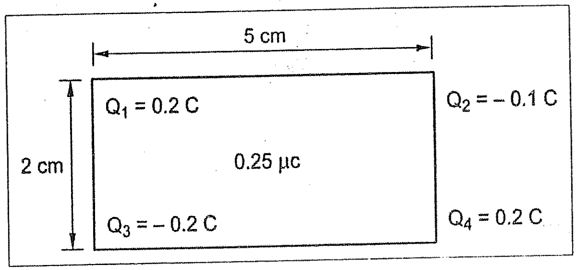
[*x* component is zero]

$$= 2 \times 0.05625 \times \frac{\sqrt{3}}{2} = 0.05625\sqrt{3} \text{ Newtons}$$

$$= \mathbf{0.09743\ Newtons}$$

☺☺

Example 1.48 *Four concentrated charges are located at the vertices of a plane rectangular shown in Fig. Find the magnitude and direction of resultant force on Q_1.*



☺***Solution:*** Force on Q_1 due to Q_2

$$F_{12} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} = \frac{(0.2)(-0.1)}{4\pi \times \frac{1}{36\pi} \times (0.05)^2}$$

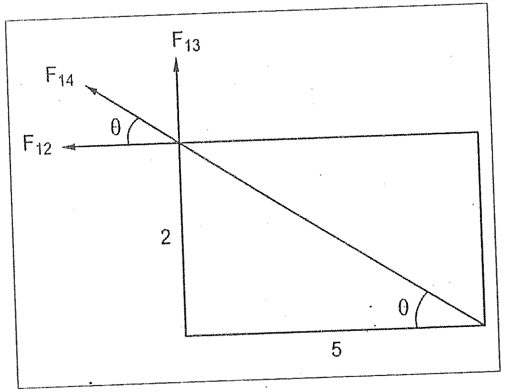
$$= -72 \text{ Newtons}$$

Force on Q_1 due to Q_3

$$F_{13} = \frac{(-0.2)(0.2)}{4\pi \times \frac{1}{36\pi} \times (0.02)^2} = -900 \text{ Newtons}$$

Force on Q_1 due to Q_4

$$F_{14} = \frac{0.2 \times 0.2 \times 10^4}{4\pi \times \frac{1}{36\pi} \times (\sqrt{2^2+5^2})} = 24.137 \text{ Newtons}$$



$$\theta = \tan^{-1} 2/5 = 21.80°$$

$$F_{14x} = F_{14} \cos\theta = 124.137 \cos 21.8° = 115.26$$

$$F_{14y} = F_{14} \sin\theta = 124.137 \sin 21.8° = 46.1$$

F_x = x components of F = $F_{12} + F_{14} \cos\theta = -72 + 115.26 = 43.26$

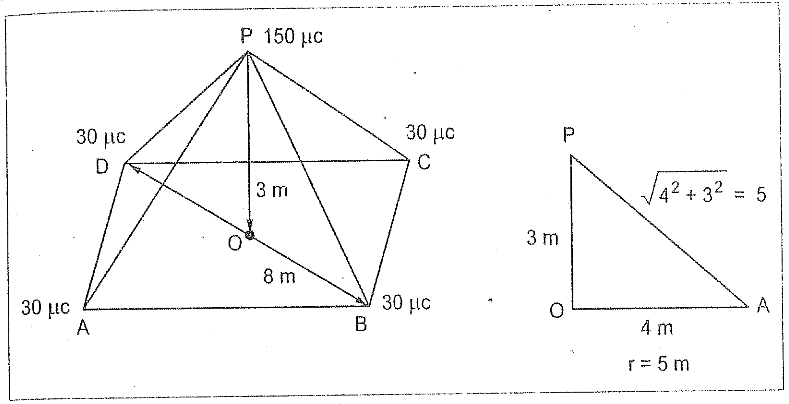
F_y = y components of F = $F_{13} + F_{14} \sin\theta = -900 + 46.1 = -853.9$

Resultant force $F = \sqrt{F_x^2 + F_y^2} \quad \tan^{-1}\left(\frac{F_y}{F_x}\right)$

$= \mathbf{855 \angle -87.1° \text{ Newtons}}$

☺☺

Example 1.49 *Four like charges of 30 μc each are located at the four corners of a square, the diagonal of which measures 8 m. Find the force on a 150 μc charge located 3 m above the centre of the square.*



☺***Solution:*** Force on 150 μc at P due to charge at A.

$$F_1 = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2} = \frac{30 \times 10^{-6} \times 150 \times 10^{-6}}{4\pi \times \frac{1}{36\pi \times 10^9} (5)^2}$$

$$= \frac{30 \times 150 \times 10^{-12} \times 9 \times 10^9}{25} = 1.62 \text{ N}$$

For on 150 μc at P due to charge at B, C, D.

$$F_2 = F_3 = F_4 = F_1 = 1.62 \text{ N}$$

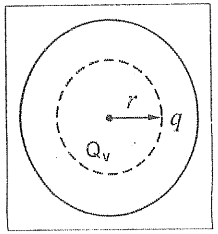
Total force at P due to all four charges

$$|F| = |F_1| + |F_2| + |F_3| + |F_4|$$

$$= 1.62 \times 4 = \mathbf{6.48 \text{ N}}$$

☺☺

Example 1.50 *A positive charge Q_v c/m³ occupies the volume of a sphere. At a point in the interior at a distance of 'r' from the centre, a small probe of charge of +q is inserted. What is the force acting on the probe charge?*



☺**Solution:** Applying Gauss's law,

The electric field at a distance 'r' from the centre is given by

$$\int_s E \cdot ds = \frac{Q}{\varepsilon_0}$$

$$\text{But}\quad \text{Charge} = \text{Volume charge density} \times \text{Volume}$$

$$\text{i.e.,}\quad Q = Q_v \times \frac{4}{3}\pi r^3$$

$$\int_s E \cdot ds = \frac{Q_v \times \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{Q_v \times \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$E = \frac{Q_v r}{3\,\varepsilon_0}$$

The force acting on the probe charge is given by $F = q\,E$.

$$\mathbf{F = q \cdot \frac{Q_v\, r}{3\,\varepsilon_0}}$$

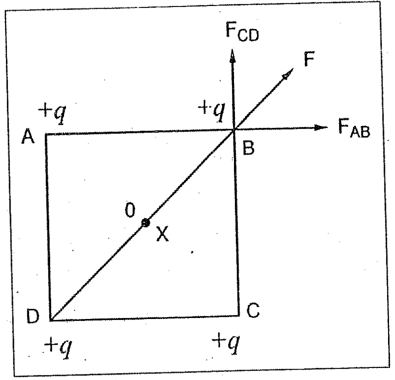
☺☺

Example 1.51 *It is required to hold four equal point charges $+q$ each in equilibrium at the corners of a square. Find the point charge which will do this, i placed at the centre of the square.*

☺**Solution:** Consider four equal positive charges are placed at the corners of a square of size 'a'. A unknown charge X is placed at the centre of the square to satisfy the equilibrium condition of four charges.

The force between two charges (at A and B) along AB, $F_{AB} = \dfrac{q^2}{4\pi\,\varepsilon\,a^2}$

The force between two charges (at B and C) along AB, $F_{BC} = \dfrac{q^2}{4\pi\,\varepsilon\,a^2}$



The force between two charges (at B and D) along AB, $F_{BD} = \dfrac{q^2}{4\pi\,\varepsilon\,(a\sqrt{2})^2}$

$$= \frac{q^2}{8\pi\,\varepsilon\,a^2}$$

The force between two charges (at O & B) along BD, $F_{BD} = \dfrac{qX}{4\pi\,\varepsilon\left(\frac{a\sqrt{2}}{2}\right)^2}$

$$= \frac{qX}{2\pi\,\varepsilon\,a^2}$$

The resultant force of F_{AB} and F_{BC} along BD $= \dfrac{q^2\sqrt{2}}{4\pi\,\varepsilon\,a^2}$

The total resultant force due to all forces along BD

$$\frac{q^2\sqrt{2}}{4\pi\,\varepsilon\,a^2} + \frac{q^2}{8\pi\,\varepsilon\,a^2} + \frac{qX}{2\pi\,\varepsilon\,a^2} = 0$$

$$\frac{q}{2\pi\,\varepsilon\,a^2}\left[\frac{q}{\sqrt{2}} + \frac{q}{4} + X\right] = 0$$

$$\frac{q}{\sqrt{2}} + \frac{q}{4} + X = 0$$

$$0.957\,q + X = 0$$

$$X = -0.957\,q$$

The required unknown charge at the centre of square is $-0.957\,q$. ☺☺

Example 1.52 *Calculate electric field intensity at P(3, –4, 2) in free space called by*

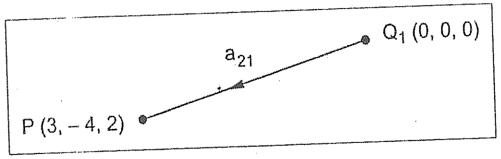
(a) $Q_1 = 2\ \mu c$ *at (0, 0, 0)*

(b) $Q_2 = 3\ \mu c$ *at (–1, 2, 3)*

(c) $Q_1 = 2\ \mu c$ *at (0, 0, 0) and* $Q_2 = 3\ \mu c$ *at (–1, 2, 3)*

☺***Solution:***

(a)



$$\text{Unit vector } \bar{a}_{21} = \frac{r_1 - r_2}{|r_1 - r_2|}$$

$$= \frac{3\,\bar{a}_x - 4\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{9+16+4}}$$

$$= \frac{3\,\bar{a}_x + 4\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{29}}$$

Electric field intensity at P due to Q_1

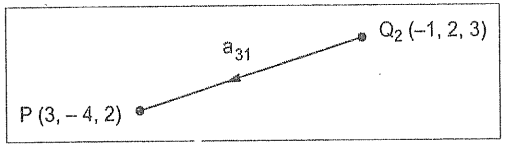
$$E = \frac{Q_1}{4\pi\,\varepsilon_0\,r^2}\,\bar{a}_{21}$$

$$= \frac{2\times 10^{-6}}{4\pi \times \dfrac{1}{36\pi\times 10^9}\times(\sqrt{29})^2}\cdot\frac{3\,\bar{a}_x + 4\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{29}}$$

$$= \frac{2\times 9\times 10^3}{29\sqrt{29}}\,(3\,\bar{a}_x + 4\,\bar{a}_y + 2\,\bar{a}_z)$$

$$\mathbf{E = 345\,\bar{a}_x - 460\,\bar{a}_y + 230\,\bar{a}_z\quad V/m}$$

(b)



$$\bar{a}_{31} = \frac{\bar{a}_x(3-(-1)) + \bar{a}_y(-4-(2)) + \bar{a}_z(2-3)}{(3+1)^2 + (-4-2)^2 + (2-3)^2}$$

$$= \frac{4\,\bar{a}_x - 6\,\bar{a}_y - \bar{a}_z}{\sqrt{16+36+1}}\cdot\frac{4\,\bar{a}_x - 6\,\bar{a}_y - \bar{a}_z}{\sqrt{53}}$$

Electric intensity at P due to Q_2

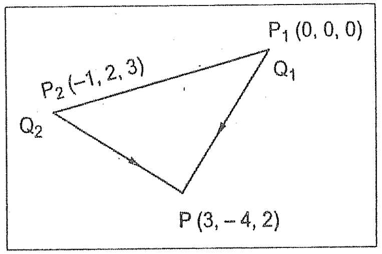
$$E = \frac{Q}{4\pi\,\varepsilon_0\,r^2}\,\bar{a}_{31}$$

$$= \frac{3\times 10^{-6}}{4\pi \times \dfrac{1}{36\pi\times 10^9}\times(\sqrt{53})^2}\cdot\frac{4\,\bar{a}_x - 6\,\bar{a}_y - \bar{a}_z}{\sqrt{53}}$$

$$= \frac{3\times 9\times 10^3}{53\sqrt{53}}\,(4\,\bar{a}_x - 6\,\bar{a}_y - \bar{a}_z)$$

$$\mathbf{E = 280\,\bar{a}_x - 420\,\bar{a}_y - 70\,\bar{a}_z\quad V/m}$$

(c)



$$\bar{a}_{31} = \frac{3\,\bar{a}_x - 4\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{9+16+4}} = \frac{3\,\bar{a}_x - 4\,\bar{a}_y + 2\,\bar{a}_z}{\sqrt{29}}$$

Electric field intensity at P due to Q_1

$$E_1 = 345\,\bar{a}_x - 460\,\bar{a}_y + 230\,\bar{a}_z \text{ V/m}$$

Electric field intensity at P due to Q_2

$$E_2 = 280\,\bar{a}_x - 420\,\bar{a}_y + 70\,\bar{a}_z \text{ V/m}$$

Total electric field intensity at P due to Q_1 and Q_2

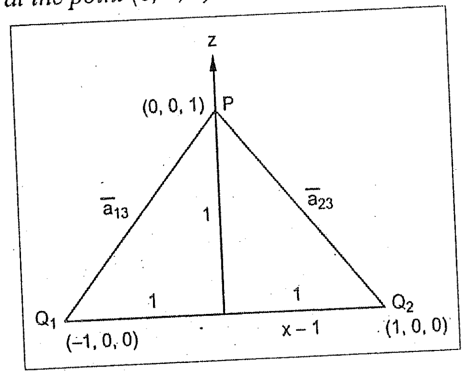
$$E = E_1 + E_2$$

$$= 345\,\bar{a}_x - 460\,\bar{a}_y + 230\,\bar{a}_z + 280\,\bar{a}_x - 420\,\bar{a}_y + 70\,\bar{a}_z$$

$$\mathbf{E = 65\,\bar{a}_x - 880\,\bar{a}_y + 160\,\bar{a}_z \text{ V/m}}$$

☺☺

Example 1.53 *Consider two point charges $Q_1 = 8\pi\varepsilon_0$ coulombs an $Q_2 = -4\pi\varepsilon_0$ coulombs situated at (–1, 0, 0) and (1, 0, 0) m respectively. Find th electric field intensity at the point (0, 0, 1).*



☺**Solution:**

$$\text{Unit vector } \bar{a}_{13} = \frac{[0-(-1)]\,\bar{a}_x + (0-0)\,\bar{a}_y + (1-0)\,\bar{a}_z}{\sqrt{1^2+0^2+1^2}}$$

$$= \frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}}$$

Electric field intensity at P due to Q_1

$$E_1 = \frac{Q_1}{4\pi\varepsilon_0 r^2}$$

$$= \frac{8\pi\varepsilon_0}{4\pi\varepsilon_0(\sqrt{2}\,)^2}\left(\frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right)$$

$$= \frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}} \text{ V/m}$$

$$\text{Unit vector } \bar{a}_{23} = \frac{(0-1)\,\bar{a}_x + (0-0)\,\bar{a}_y + (1-0)\,\bar{a}_z}{\sqrt{1^2+0+1^2}}$$

$$= \frac{-\bar{a}_x + \bar{a}_z}{\sqrt{2}}$$

Electric field intensity at P due to Q_2

$$E_2 = \frac{Q_2}{4\pi\varepsilon_0 r^2}$$

$$= \frac{-4\pi\varepsilon_0}{4\pi\varepsilon_0(\sqrt{2}\,)^2}\left(\frac{-\bar{a}_x + \bar{a}_z}{\sqrt{2}}\right) = \frac{-\bar{a}_x + \bar{a}_z}{2\sqrt{2}} \text{ V/m}$$

Total electric field at P due to Q_1 and Q_2

$$E = E_1 + E_2 = \frac{\bar{a}_x + \bar{a}_z}{\sqrt{2}} - \frac{\bar{a}_x + \bar{a}_z}{2\sqrt{2}}$$

$$= \frac{2\,\bar{a}_x + 2\,\bar{a}_z - \bar{a}_x + \bar{a}_z}{2\sqrt{2}}$$

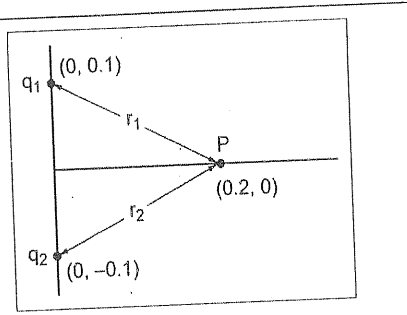
$$= \frac{\bar{a}_x + 3\,\bar{a}_z}{2\sqrt{2}} \text{ V/m}$$

☺☺

Example 1.54 *A positive point charge 100×10^{-12} C is located in air at x = 0, y = 0.1 m and another charge at x = 0, y = – 0.1 m. What is the magnitude and the direction of E at x = 0.2 m, y = 0?*

☺**Solution:** $q_1 = 100 \times 10^{-12}$ C at (0, 0.1) m

$q_2 = 100 \times 10^{-12}$ C at (0, – 0.1) m



$$r_1 = \sqrt{(0.2-0)^2+(0-0.1)^2} = 0.22 \text{ m}$$

$$r_2 = \sqrt{(0.2-0)^2+(0-(-0.1))^2} = 0.22 \text{ m}$$

$$\bar{a}_{r_1} = \frac{0.2\,\bar{a}_x - 0.1\,\bar{a}_y}{\sqrt{(0.2)^2+(0.1)^2}} = \frac{0.2\,\bar{a}_x - 0.1\,\bar{a}_y}{0.22}$$

$$\bar{a}_{r_2} = \frac{0.2\,\bar{a}_x + 0.1\,\bar{a}_y}{\sqrt{(0.2)^2+(0.1)^2}} = \frac{0.2\,\bar{a}_x + 0.1\,\bar{a}_y}{0.22}$$

Electric field at P due to q_1

$$E_1 = \frac{q_1}{4\pi\,\varepsilon_0\,r_1^2}\,\bar{a}_{r_1}$$

$$= \frac{100\times 10^{-12}}{4\pi\times 8.854\times 10^{-12}\times(0.22)^2}\left(\frac{0.2\,\bar{a}_x - 0.1\,\bar{a}_y}{0.22}\right)$$

$$= 16.88\,\bar{a}_x - 8.44\,\bar{a}_y \text{ V/m}$$

Electric field at P due to q_2

$$E_2 = \frac{q_2}{4\pi\,\varepsilon_0\,r_2^2}\,\bar{a}_{r_2}$$

$$= \frac{100\times 10^{-12}}{4\pi\times 8.854\times 10^{-12}\times(0.22)^2}\left(\frac{0.2\,\bar{a}_x + 0.1\,\bar{a}_y}{0.22}\right)$$

$$= 16.88\,\bar{a}_x + 8.44\,\bar{a}_y \text{ V/m}$$

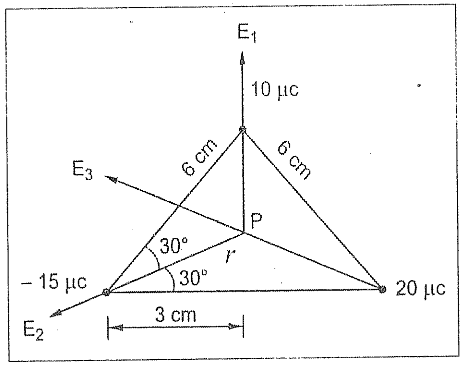
Total electric field at P due to q_1 and q_2

$$E = E_1 + E_2 = 16.88\,\bar{a}_x - 8.44\,\bar{a}_y + 16.88\,\bar{a}_x + 8.44\,\bar{a}_y$$

$$= \mathbf{33.76\,\bar{a}_x \text{ V/m}}$$

☺☺

Example 1.55 *Charges of 10 μc, 20 μc, and – 15μc are placed at the corners of an equilateral triangle which has sides 6 cm long. The dielectric constant of the medium is 3. Find the electric field intensity at the centre of the triangle equidistant from all the three charges.*



☺***Solution:*** From the Fig.,

$$r = \frac{3}{\cos\theta} = \frac{3}{\cos 30^\circ} = 3.46 \text{ cm}$$

Electric field intensity due to 10 μc charge at P

$$E_1 = \frac{q}{4\pi\,\varepsilon_0\,r^2} = \frac{10\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times(3.46\times 10^{-2})^2}$$

$$= 25\times 10^6 \text{ V/m}$$

Electric field intensity due to – 15 μc charge.

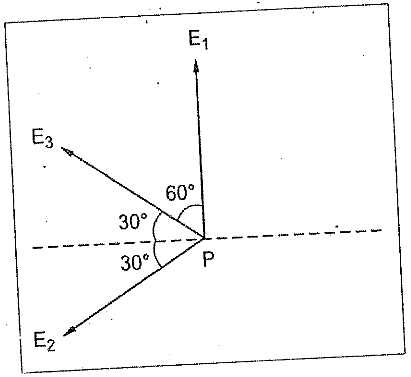
$$E_2 = \frac{-15\times 10^{-6}}{4\pi\times 8.854\times 10^{-12}\times(3.46\times 10^{-2})^2}$$

$$= -37.5\times 10^6 \text{ V/m}$$

Electric field intensity due to 20 μc charge

$$E_3 = \frac{20 \times 10^{-6}}{4\pi \times 8.854 \times 10^{-12} \times (3.46 \times 10^{-2})^2}$$

$$= 50 \times 10^6 \text{ V/m}$$



$$E_1 = 0 + \bar{a}_y\ 25 \times 10^6$$

$$E_2 = E_2 \cos\theta\, \bar{a}_x + E_2 \sin\theta\, \bar{a}_y$$

$$= -32.48 \times 10^6\, \bar{a}_x - 18.75 \times 10^6\, \bar{a}_y \quad \text{V/m}$$

$$E_3 = E_3 \cos\theta\, \bar{a}_x + E_3 \sin\theta\, \bar{a}_y$$

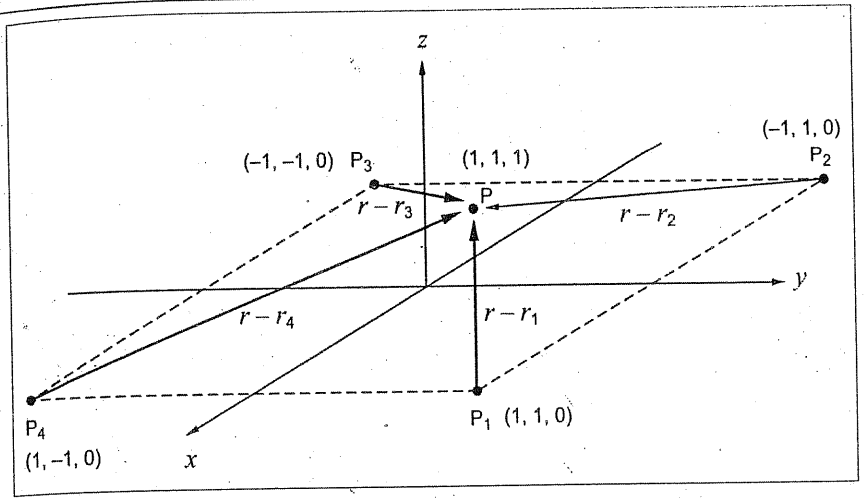
$$= 43.3 \times 10^6\, \bar{a}_x + 25 \times 10^6\, \bar{a}_y \quad \text{V/m}$$

$$E = E_1 + E_2 + E_3 = 21.65 \times 10^6 \quad \text{V/m}$$

$$= (25\, \bar{a}_y - 32.48\, \bar{a}_x - 18.75 \times 10^6\, \bar{a}_y + 43.3\, \bar{a}_x + 25 \times 10^6\, \bar{a}_y)\, 10^6 \quad \text{V/m}$$

$$= \mathbf{(10.82\, \bar{a}_x + 31.25\, \bar{a}_y) \times 10^6 \quad V/m}$$

Example 1.56 *Find the electric field intensity produced by a point charg distribution at P(1, 1, 1) caused by four identical 3 nc point charges located a P₁(1, 1, 0), P₂(–1, 1, 0), P₃(–1, –1, 0) and P₄ (1, –1, 0).*



☺**Solution:** Electric field intensity at P due to point charge 3 *nc* at P₁

$$E_1 = \frac{Q}{4\pi\varepsilon_0} \frac{1}{r^2} \bar{a}_{12}$$

$$E_1 = \frac{Q}{4\pi\varepsilon_0 r^2} \left[\frac{(1-1)\,\bar{a}_x + (1-1)\,\bar{a}_y + (1-0)\,\bar{a}_z}{\sqrt{1}} \right]$$

$$E_1 = \frac{Q}{4\pi\varepsilon_0} \bar{a}_z \quad \text{V/m} \qquad r = 1$$

Electric field intensity at P due to point charge 3 *nc* at P₂

$$E_2 = \frac{Q}{4\pi\varepsilon_0 r^2} \left[\frac{(1+1)\,\bar{a}_x + (1-1)\,\bar{a}_y + (1-0)\,\bar{a}_z}{\sqrt{2^2+1}} \right]$$

$$= \frac{Q}{4\pi\varepsilon_0} \left[\frac{2\,\bar{a}_x + \bar{a}_z}{(\sqrt{5})^3} \right] \quad \text{V/m}$$

Electric field intensity at P due to point charge 3 *nc* at P_3

$$E_3 = \frac{Q}{4\pi\varepsilon_0 r^2}\left[\frac{(1+1)\bar{a}_x + (1+1)\bar{a}_y + (1-0)\bar{a}_z}{\sqrt{2^2+2^2+1}}\right]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{2\bar{a}_x + 2\bar{a}_y + \bar{a}_z}{(3)^3}\right] \text{ V/m}$$

Electric field intensity at P due to point charge 3 *nc* at P_4

$$E_4 = \frac{Q}{4\pi\varepsilon_0 r^2}\left[\frac{(1-1)\bar{a}_x + (1+1)\bar{a}_y + (1-0)\bar{a}_z}{\sqrt{2^2+1}}\right] \text{ V/m}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{2\bar{a}_y + \bar{a}_z}{(\sqrt{5})^3}\right] \text{ V/m}$$

Electric field intensity at P due to all four charges

$$E = E_1 + E_2 + E_3 + E_4$$

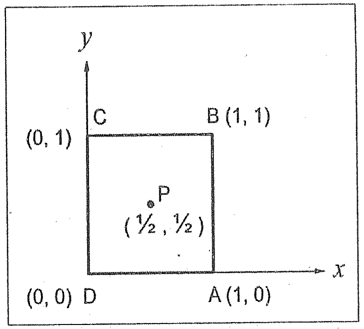
$$= \frac{Q}{4\pi\varepsilon_0}\left[\bar{a}_z + \frac{2\bar{a}_x + \bar{a}_z}{(\sqrt{5})^3} + \frac{2\bar{a}_x + 2\bar{a}_y + \bar{a}_z}{(3)^3} + \frac{2\bar{a}_y + \bar{a}_z}{(\sqrt{5})^3}\right]$$

$$= \frac{3\times10^{-9}}{4\pi\times\frac{1}{36\pi\times10^9}}\left[\begin{array}{l}\bar{a}_z + 0.179\,\bar{a}_x + 0.0894\,\bar{a}_z + 0.074\,\bar{a}_x + \\ 0.074\,\bar{a}_y + 0.037\,\bar{a}_z + 0.179\,\bar{a}_y + 0.0894\,\bar{a}_z\end{array}\right]$$

$$= 27\,[0.2534\,\bar{a}_x + 0.2534\,\bar{a}_y + 1.216\,\bar{a}_z]$$

$$= \mathbf{6.831\,\bar{a}_x + 6.831\,\bar{a}_y + 32.832\,\bar{a}_z} \textbf{ V/m}$$

Example 1.57 *Four positive point charges 10^{-9} each are situated in x plane at points (0, 0), (0, 1), (1, 1) and (1, 0) m. Find the electric field intensity (½ , ½).*



Solution: Electric field intensity at P due to A,

$$E_1 = \frac{Q}{4\pi\varepsilon_0 r^2}\,\bar{a}_{12}$$

$$= \frac{10^{-9}}{4\pi\times\frac{1}{36\pi\times10^9}\,r^2}\left[\frac{\left(1-\frac{1}{2}\right)\bar{a}_x + \left(0-\frac{1}{2}\right)\bar{a}_y}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\right]$$

$$= \frac{10^{-9}\times9\times10^9}{r^2}\left[\frac{\frac{1}{2}\bar{a}_x - \frac{1}{2}\bar{a}_y}{\frac{1}{\sqrt{2}}}\right]$$

$$= \frac{9}{\left[\frac{1}{\sqrt{2}}\right]^3}\left[\frac{\bar{a}_x}{2} - \frac{\bar{a}_y}{2}\right] \qquad \left[\because r = \frac{1}{\sqrt{2}}\right]$$

$$E_1 = \frac{25.456}{2}\,[\bar{a}_x - \bar{a}_y] = 12.728\,[\bar{a}_x - \bar{a}_y]$$

Electric field intensity at P due to B,

$$E_2 = \frac{9\times10^{-9}\times10^9}{r^2}\left[\frac{\left(1-\frac{1}{2}\right)\bar{a}_x + \left(1-\frac{1}{2}\right)\bar{a}_y}{\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\right] \qquad \left[\because r = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{9}{\left[\frac{1}{\sqrt{2}}\right]^3}\left[\frac{1}{2}\bar{a}_x + \frac{1}{2}\bar{a}_y\right]$$

$$E_2 = \frac{25.456}{2}\,[\bar{a}_x + \bar{a}_y] = 12.728\,[\bar{a}_x + \bar{a}_y]$$

Electric field intensity at P due to C

$$E_3 = \frac{9\times10^{-9}\times10^{9}}{r^2}\left[\frac{\left(0-\frac{1}{2}\right)\bar{a}_x + \left(1-\frac{1}{2}\right)\bar{a}_y}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}}\right]$$

$$= \frac{9}{\left(\frac{1}{\sqrt{2}}\right)^3}\left[-\frac{1}{2}\bar{a}_x + \frac{1}{2}\bar{a}_y\right] = \frac{25.456}{2}\,[-\bar{a}_x + \bar{a}_y]$$

$$E_3 = 12.728\,[-\bar{a}_x + \bar{a}_y]$$

Electric field intensity at P due to D

$$E_4 = \frac{9\times10^{-9}\times10^{9}}{r^2}\left[\frac{\left(0-\frac{1}{2}\right)\bar{a}_x + \left(0-\frac{1}{2}\right)\bar{a}_y}{\sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}}\right]$$

$$= \frac{9}{\left(\frac{1}{\sqrt{2}}\right)^3}\left[-\frac{1}{2}\bar{a}_x - \frac{1}{2}\bar{a}_y\right] = \frac{25.456}{2}\,[-\bar{a}_x - \bar{a}_y]$$

$$E_3 = 12.728\,[-\bar{a}_x - \bar{a}_y]$$

Electric field intensity at P due to all four charges

$$E = E_1 + E_2 + E_3 + E_4$$

$$= 12.728\,[\bar{a}_x - \bar{a}_y + \bar{a}_x + \bar{a}_y - \bar{a}_x + \bar{a}_y - \bar{a}_x - \bar{a}_y]$$

$$= 12.749\,[0]$$

$$\mathbf{E = 0 \ \ V/m}$$

Example 1.58 *A spherical region has volume charge density distribution as follows.*

$$\rho_v = \rho_\theta\left(1 - \frac{r^2}{a^2}\right) \ c/m^3 \ for \ r \le a$$

$$= 0 \quad r > a$$

Find the electric field intensity in all region. Show that maximum field intensity occurs at a distance of $r = 0.745\,a$ from centre of the space.

☺***Solution:*** The charge Q is given by

$$Q = \iiint_v \rho\, dv = \int dq$$

$$dv = 4\pi r^2\, dr$$

$$Q = \int_{r=0}^{a} \rho_0\left(1 - \frac{r^2}{a^2}\right) 4\pi r^2\, dr$$

$$= 4\pi\,\rho_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r^2\, dr$$

$$= 4\pi\,\rho_0\left[\frac{r^3}{3} - \frac{r^5}{5a^2}\right]_0^a$$

$$= 4\pi\,\rho_0\,\frac{2a^3}{15} = \frac{8\pi\,\rho_0\,a^3}{15} \ \text{coulombs}$$

Electric field intensity inside the sphere

$$E = \frac{Q}{4\pi\,\varepsilon_0 r^2} \quad r \le a$$

$$= \frac{1}{4\pi\,\varepsilon_0 r^2}\left[\int 4\pi\,\rho_0\left(1 - \frac{r^2}{a^2}\right) r^2 \cdot dr\right]$$

$$= \frac{\rho_0 r^2}{\varepsilon_0 r^2}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]$$

$$= \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]$$

Electric field intensity at any point outside the sphere

$$E = \frac{Q}{4\pi \varepsilon_0 r^2}$$

$$\text{But } Q = \frac{8}{15} \pi \rho_0 a^3$$

$$E = \frac{\frac{8}{15} \pi \rho_0 a^3}{4\pi \varepsilon_0 r^2}$$

$$E = \frac{2 \rho_0 a^3}{15 \varepsilon_0 r^2}$$

The maximum value of E is evaluated by

$$\frac{dE}{dr} = 0$$

$$E = \frac{\rho_0}{\varepsilon_0} \left[\frac{r}{3} - \frac{r^3}{5 a^2} \right]$$

$$\frac{dE}{dr} = \frac{\rho_0}{\varepsilon_0} \left[\frac{1}{3} - \frac{3 r^2}{5 a^2} \right] = 0$$

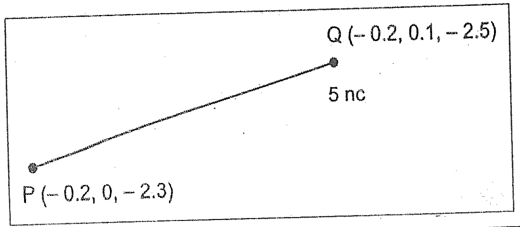
$$\frac{1}{3} - \frac{3 r^2}{5 a^2} = 0$$

$$\frac{r^2}{a^2} = \frac{5}{9}$$

$$\mathbf{r = 0.745\, a}$$

☺

Example 1.59 *Determine the electric field intensity at P(– 0.2, 0, –2.3) d to a point charge of 5 nc at Q (0.2, 0.1, –2.5) in air. All dimensions are meters.*



☺***Solution:***

$$r_P = -0.2\, \bar{a}_x - 2.3\, \bar{a}_z$$

$$r_Q = 0.2\, \bar{a}_x + 0.1\, \bar{a}_y - 2.5\, \bar{a}_z$$

$$r = r_P - r_Q$$

$$= (-0.2-0.2)\, \bar{a}_x - 0.1\, \bar{a}_y + (-2.3+2.5)\, \bar{a}_z$$

$$= -0.4\, \bar{a}_x - 0.1\, \bar{a}_y + 0.2\, \bar{a}_z$$

$$|r| = \sqrt{(0.4)^2 + (0.1)^2 + (0.2)^2} = 0.458$$

$$\text{Unit vector } \bar{r} = \frac{r}{|r|} = \frac{-(0.4)\, \bar{a}_x - (0.1)\, \bar{a}_y + (0.2)\, \bar{a}_z}{0.458}$$

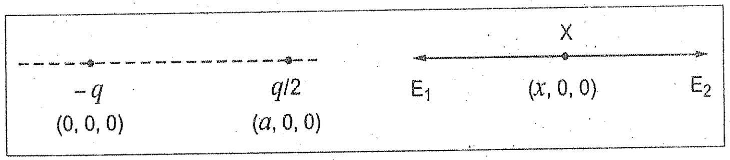
Electric field at P,

$$E = \frac{Q}{4\pi \varepsilon_0 |r|^2} \bar{r}$$

$$= \frac{5 \times 10^{-9}}{4\pi \times \frac{1}{36\pi \times 10^9} \times (0.458)^2} \left[\frac{-0.4\, \bar{a}_x - 0.1\, \bar{a}_y + 0.2\, \bar{a}_z}{0.458} \right]$$

$$= \mathbf{-186.82\, \bar{a}_x - 46.65\, \bar{a}_y + 93.52\, \bar{a}_z \ \ V/m}$$ ☺☺

Example 1.60 *A point charge of '– q' coulomb is placed upon the origin of a rectangular co-ordinate system. A second point charge of 'q/2' c is placed at a distance 'a' unit along the x-direction. At what point along the axis of the set up will there be zero electric field.*



☺***Solution:*** Assuming a point x with co-ordinates $(x, 0, 0)$ where positive test charge is placed. The magnitude of test charge is assumed to be unity.

$$|E_1| = \frac{-q}{4\pi\,\varepsilon\,x^2}$$

$$|E_2| = \frac{+q/2}{4\pi\,\varepsilon\,(a-x)^2}$$

Now at point x, $|E_1| + |E_2| = 0$

$$\frac{-q}{4\pi\,\varepsilon\,x^2} + \frac{q/2}{4\pi\,\varepsilon\,(x-a)^2} = 0$$

$$\frac{-1}{x^2} + \frac{1}{2\,(x-a)^2} = 0$$

$$x^2 = 2x^2 + 2a^2 - 4ax$$

$$x^2 - 4ax + 2a^2 = 0$$

$$x = \frac{4a \pm \sqrt{16a^2 - 8a^2}}{2}$$

$$= \frac{4a \pm 2\sqrt{2}\,a}{2}$$

$$= (2 \pm \sqrt{2})\,a$$

∴ The co-ordinates of the point x are $((2+\sqrt{2})\,a, 0, 0)$. The co-ordinates cannot be $(2-\sqrt{2})\,a$, since, it lies between '$+q/2$' and '$-q$', where the charge (unit positive) will accelerate towards '$-q$'. ☺☺

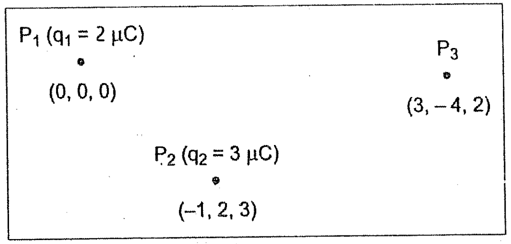
Example 1.61 *In a rectangular co-ordinate referenced space two point charges are placed such that, a charge of 2 μc is placed at the origin and the second point charge of 3 μc at (–1, 2, 3).*

Determine,

(i) The vectorial electric field at a point (+3, –4, 2) due to 2 μc charge.

(ii) The vectorial electric field at a point (3, –4, 2) due to 3 μc charge.

(iii) The equivalent electric field at (3, –4, 2) if the two charges are considerable isolated from external fields.



☺**Solution:**

(i) $$E_1 \text{ at } P_3 = \frac{q_1\,\bar{a}_1}{4\pi\,\varepsilon_0\,r_1^2} = \frac{2\,\mu(-3\,\bar{a}_x + 4\,\bar{a}_y - 2\,\bar{a}_z)}{4\pi\,(8.854\times10^{-12})\,(3^2+4^2+2^2)^{3/2}} \text{ V/m}$$

$$= \mathbf{115.1\,(-3\,\bar{a}_x + 4\,\bar{a}_y - 2\,\bar{a}_z)\ \ V/m}$$

(ii) $$E_2 \text{ at } P_3 = \frac{q_2\,\bar{a}_2}{4\pi\,\varepsilon_0\,r_2^2} = \frac{3\,\mu(-4\,\bar{a}_x + 6\,\bar{a}_y + \bar{a}_z)}{4\pi\,(8.854\times10^{-12})\,(4^2+6^2+1^2)^{3/2}}$$

$$= \mathbf{95.625\,(-4\,\bar{a}_x + 6\,\bar{a}_y + \bar{a}_z)\ \ V/m}$$

(iii) The effect of two fields at a point can be determined by vectorially adding the two field intensities.

$$E_{total} = E_1 + E_2$$

$$= 115.1\,(-3\,\bar{a}_x + 4\,\bar{a}_y - 2\,\bar{a}_z) + 95.625\,(-4\,\bar{a}_x + 6\,\bar{a}_y + \bar{a}_z) \text{ V/m}$$

$$= \mathbf{-727.8\,\bar{a}_x + 1034.15\,\bar{a}_y - 134.5745\,\bar{a}_z\ \ V/m}$$ ☺☺

Example 1.62 *Determine the E field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_0$ for $0 \le r \le a$ and $\rho = 0$ for $r > a$. Show the plot of E versus r.*

☺**Solution:** Consider a spherical cloud of electrons of radius 'a' with a volume charge density.

$$\rho = -\rho_0 \qquad 0 \le r \le a$$

$$= 0 \qquad \text{for } r > a$$

The charge enclosed by the spherical cloud of electrons is given by

$$Q = \iiint_v \rho \; dv$$

$$dQ = -\rho_0 \cdot 4\pi r^2 \cdot dr$$

$$Q = -4\pi \rho_0 \int r^2 \cdot dr = -4\pi \rho_0 \frac{r^3}{3}$$

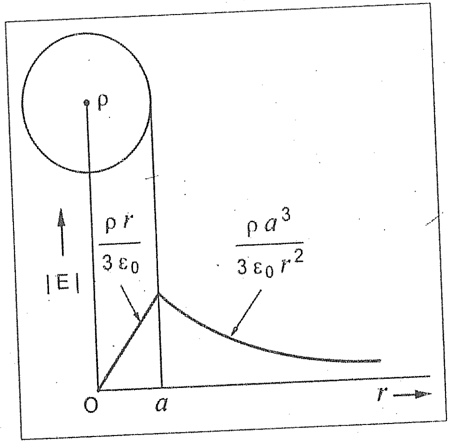
$$= \frac{-4}{3} \pi \rho_0 r^3$$

By applying Gauss's law inside the spherical cloud

$$\oint D \cdot ds = Q$$

$$\oint E \cdot ds = \frac{Q}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} = \frac{-4\pi \rho_0 r^3}{3 \varepsilon_0}$$


ρ
$\frac{\rho r}{3\varepsilon_0}$
$\frac{\rho a^3}{3\varepsilon_0 r^2}$
|E|
O
a
r

The electric field intensity inside the spherical cloud, $0 \le r \le a$

$$E = \frac{-4\pi \rho_0 r^3}{3 \varepsilon_0 4\pi r^2}$$

$$E = \frac{-\rho_0 r}{3 \varepsilon_0}$$

$$|E| = \frac{\rho_0 r}{3 \varepsilon_0}$$

The charge just outside cloud $r > a$ is zero. However, the electric field intensity at any distance $r > a$ due to charge enclosed by the spherical cloud is given by

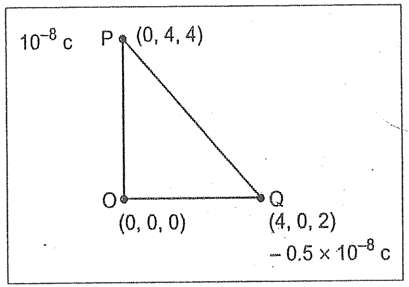
$$E = \frac{Q}{4\pi r^2 \varepsilon_0}$$

But $Q = -\frac{4}{3} \pi \rho_0 a^3 \quad (r = a)$

$$E = \frac{-4\pi \rho_0 a^3}{3 \times 4\pi r^2 \varepsilon_0} = \frac{-\rho_0 a^3}{3 \varepsilon_0 r^2}$$

$$|E| = \frac{\rho_0 a^3}{3 \varepsilon_0 r^2}$$ ☺☺

Example 1.63 *Find the total electric field at the origin due to 10^{-8} c charge located at $P(0, 4, 4)$ m and -0.5×10^{-8} c charge at $Q(4, 0, 2)$ m.*



☺ ***Solution:***

$$\overline{OP} = \frac{(0-0)\,\bar{a}_x + (0-4)\,\bar{a}_y + (0-4)\,\bar{a}_z}{\sqrt{0 + (0-4)^2 + (0-4)^2}}$$

$$= \frac{-4\,\bar{a}_y - 4\,\bar{a}_z}{4\sqrt{2}} = \frac{-\bar{a}_y - \bar{a}_z}{\sqrt{2}}$$

$$\overline{OQ} = \frac{(0-4)\,\bar{a}_x + 0 + (0-2)\,\bar{a}_z}{\sqrt{(0-4)^2 + 0 + (0-2)^2}} = \frac{-4\,\bar{a}_x - 2\,\bar{a}_z}{2\sqrt{5}}$$

$$= \frac{-2\,\bar{a}_x - \bar{a}_z}{\sqrt{5}}$$

Electric field at O due to P

$$E_1 = \frac{Q}{4\pi\varepsilon_0 r^2}\,\overline{OP} = \frac{10^{-8}}{4\pi \times 8.854 \times 10^{-12}(\sqrt{2})^2} \cdot \frac{(-\bar{a}_y - \bar{a}_z)}{\sqrt{2}}$$

$$= 31.79\,(-\bar{a}_y - \bar{a}_z)$$

Electric field at O due to Q

$$E_2 = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{-0.5 \times 10^{-8}}{4\pi \times 8.854 \times 10^{-12} \times (\sqrt{5})^2}\;\frac{(-2\,\bar{a}_x - \bar{a}_z)}{\sqrt{5}}$$

$$= 4.02\,(-2\,\bar{a}_x - \bar{a}_z)$$

Total electric field at O,

$$E = E_1 + E_2$$

$$= 31.79\,(-\bar{a}_y - \bar{a}_z) + 4.02\,(-2\,\bar{a}_x - \bar{a}_z)$$

$$= -(\mathbf{8.043\,\bar{a}_x + 31.79\,\bar{a}_y + 35.81\,\bar{a}_z})$$ ☺☺

Example 1.64 *Find E in cartesian coordinates at the point (1, 2, 3) in fre space if the source of the field is*

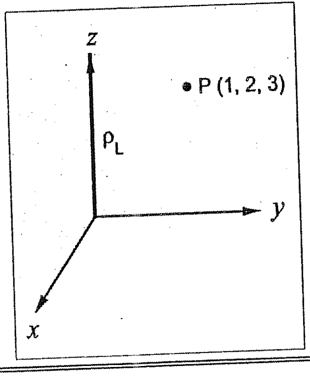
(a) a line charge of $\rho_L = 10^{-9}$ c/m along the z axis.

(b) a line charge of $\rho_L = 10^{-9}$ c/m along the y axis.

☺ ***Solution:*** ***(a) $\rho_L = 10^{-9}$ c/m***

Line charge ρ_L is along the z axis. Consider a point (0, 0, z) in the line charge.

Electric field at P(1, 2, 3) is



$$E = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a}_r$$

$$\bar{r} = (1-0)\,\bar{a}_x + (2-0)\,\bar{a}_y = \bar{a}_x + 2\,\bar{a}_y$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{\bar{a}_x + 2\,\bar{a}_y}{\sqrt{1+4}} = \frac{\bar{a}_x + 2\,\bar{a}_y}{\sqrt{5}}$$

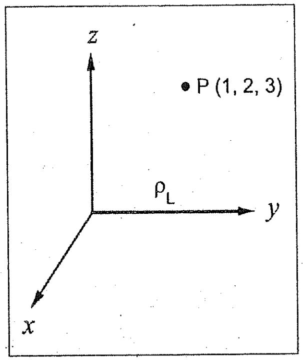
$$\therefore\; E = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a}_r$$

$$= \frac{10^{-9}\,(\bar{a}_x + 2\,\bar{a}_y)}{2\pi \times 8.854 \times 10^{-12} \times 5}$$

$$= \mathbf{3.596\,(\bar{a}_x + 2\,\bar{a}_y)}$$

(b) Line charge ρ_L is along the y axis.

Consider a point (0, y, 0) on the line charge.



Electric field at P(1, 2, 3) is

$$E = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a}_r$$

$$\bar{r}' = (1-0)\,\bar{a}_x + (3-0)\,\bar{a}_z = \bar{a}_x + 3\,\bar{a}_z$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|}$$

$$= \frac{\bar{a}_x + 3\,\bar{a}_z}{\sqrt{1+9}} = \frac{\bar{a}_x + 3\,\bar{a}_z}{\sqrt{10}}$$

$$\therefore\; E = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\bar{a}_r$$

$$= \frac{10^{-9}\,(\bar{a}_x + 3\,\bar{a}_z)}{2\pi \times 8.854 \times 10^{-12} \times 10}$$

$$= \mathbf{1.797\,(\bar{a}_x + 3\,\bar{a}_z)}$$ ☺☺

Example 1.65 *A circular disc of 10 cm radius is charged uniformly with a total charge of 10^{-6} c. Find the electric intensity at a point 30 cm away from the disc along the axis.*

☺**Solution:**

$$\rho_s = \frac{Q}{A} = \frac{10^{-6}}{\pi r^2} = \frac{10^{-6}}{\pi \times (0.1)^2} = \frac{10^{-4}}{\pi}$$

$$E = \frac{\rho_s}{2\varepsilon}\left[1 - \frac{h}{\sqrt{h^2+R^2}}\right]$$

$$= \frac{10^{-4}/\pi}{2 \times \frac{1}{36\pi \times 10^9}}\left[1 - \frac{0.3}{\sqrt{0.3+0.1}}\right]$$

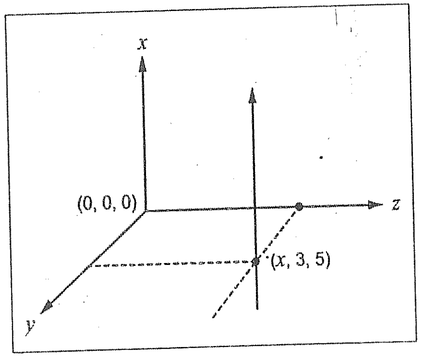
$$= 18 \times 10^5\left[1 - \frac{0.3}{0.63}\right] = 18 \times 10^3\,[1 - 0.467]$$

$$= \mathbf{942.85\ kV/m}$$

☺☺

Example 1.66 *An infinitely long uniform line charge is located at $y = 3$, $z = 5$. If $\rho_l = 30$ nc/m, find field intensity at: (i) Origin (ii) P(0, 6, 1) and (iii) P(5, 6, 1).*

☺**Solution:**



(i) **$\bar{E}$ at (0, 0, 0)**

$$\bar{r} = (0-3)\,\bar{a}_y + (0-5)\,\bar{a}_z = -3\,\bar{a}_y - 5\,\bar{a}_z$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{-[3\,\bar{a}_y + 5\,\bar{a}_z]}{\sqrt{3^2+5^2}} = \frac{-[3\,\bar{a}_y + 5\,\bar{a}_z]}{\sqrt{34}}$$

$\bar{E}$ due to an infinite line charge

$$\bar{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\,\bar{a}_r = \frac{-30 \times 10^{-9}}{2\pi\varepsilon_0 \times \sqrt{34}} \times \frac{3\,\bar{a}_y + 5\,\bar{a}_z}{\sqrt{34}}$$

$$= -(47.58\,\bar{a}_y + 79.3\,\bar{a}_z)\ \text{V/m}$$

(ii) **$\bar{E}$ at (0, 6, 1)**

$$\bar{r} = (6-3)\,\bar{a}_y + (1-5)\,\bar{a}_z = 3\,\bar{a}_y - 4\,\bar{a}_z$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{3\,\bar{a}_y - 4\,\bar{a}_z}{\sqrt{3^2+4^2}} = \frac{3\,\bar{a}_y - 4\,\bar{a}_z}{5}$$

$$\bar{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\,\bar{a}_r = \frac{30 \times 10^{-9}}{2\pi\varepsilon_0 5^2}\cdot(3\,\bar{a}_y - 4\,\bar{a}_z)$$

$$= 64.7\,\bar{a}_y - 86.2\,\bar{a}_z\ \text{V/m}$$

(iii) **$\bar{E}$ at (5, 6, 1)**

$$\bar{r} = (5-0)\,\bar{a}_x + (6-3)\,\bar{a}_y + (1-5)\,\bar{a}_z$$

$$= 5\,\bar{a}_x + 3\,\bar{a}_y - 4\,\bar{a}_z$$

$$\bar{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{5\,\bar{a}_x + 3\,\bar{a}_y - 4\,\bar{a}_z}{\sqrt{5^2+3^2+4^2}} = \frac{5\,\bar{a}_x + 3\,\bar{a}_y - 4\,\bar{a}_z}{\sqrt{50}}$$

$$\bar{E} = \frac{30 \times 10^{-9}}{2\pi\varepsilon_0(\sqrt{50})^2}\cdot(5\,\bar{a}_x + 3\,\bar{a}_y - 4\,\bar{a}_z)$$

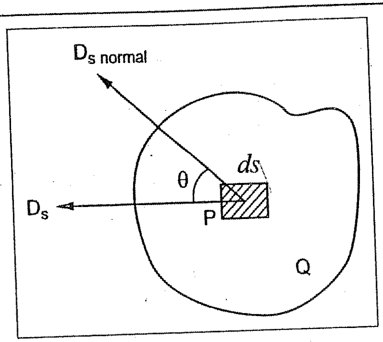
$$= \mathbf{(53.93\,\bar{a}_x + 32.36\,\bar{a}_y - 43.14\,\bar{a}_z)\ V/m}$$

☺☺

Example 1.67 *State Gauss law for the electric fields. Derive its integral and ifferential forms.*

☺**Solution:**

Gauss's law for electric field: The electric flux passing through any closed urface is equal to the total charge enclosed by the surface.



Consider a small element ds in a plane surface having a charge Q and P be point in the element ds. Let D_s make an angle θ with ds as shown in Fig.

The flux crossing ds is then the product of the normal component of D_s and ds.

$$d\chi = D_s\,\text{normal} \cdot ds = D_s \cos\theta \cdot ds$$

$$d\chi = D_s \cdot ds$$

The total flux passing through the closed surface is given by

$$\chi = \int d\chi = \oint_s D_s \cdot ds$$

$$\chi = Q$$

In terms of volume of charge density,

$$\chi = \int_v \rho_v \; dv = Q$$

$$\int_v D \cdot ds = \int_v \rho \cdot dv = Q$$

This is the integral form of electric Gauss's law.

By applying divergence theorem

$$\int_s D \cdot ds = \int_v \nabla \cdot D \; dv$$

Comparing the above two equations,

$$\int_v \nabla \cdot D \; dv = \int_v \rho \cdot dv$$

$$\therefore \; \nabla \cdot D = \rho$$

This is the differential form of electric Gauss law.

Example 1.68 *Obtain the expressions for D and E using Gauss's law.*

Solution: Gauss law, $Q = \oint_s D \cdot ds$

Let r be the radius of sphere, D be the flux density in radial direction a_r.

$$\text{Let } |D| = D_r$$

$$D = D_r \, \bar{a}_r$$

Let the Gaussian surface is a sphere of radius r enclosing charge Q.

For a Gaussian surface *i.e.*, sphere of radius r, ds normal to $\bar{a}_r$ is

$$ds = r^2 \sin\theta \cdot d\theta \cdot d\phi \cdot \bar{a}_r$$

$$D \cdot ds = D \cdot r^2 \sin\theta \cdot d\theta \cdot d\phi$$

$$\oint D \cdot ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r \; r^2 \sin\theta \cdot d\theta \cdot d\phi$$

$$= \int_{\theta=0}^{\pi} D_r \; r^2 \sin\theta \cdot d\theta \, [\,\phi\,]_0^{2\pi}$$

$$= \int_{\theta=0}^{\pi} 2\pi \; D_r \; r^2 \sin\theta \cdot d\theta$$

$$= 2\pi D_r r^2 \, [-\cos\theta\,]_0^{\pi}$$

$$= 4\pi r^2 \; D_r$$

$$\oint_s D \cdot ds = Q = 4\pi r^2 \; D_r$$

$$\therefore \quad D_r = \frac{Q}{4\pi r^2}$$

$$D = D_r \bar{a}_r = \frac{Q}{4\pi r^2} \cdot \bar{a}_r$$

$$E = \frac{D}{\varepsilon_0} = \frac{Q}{4\pi \varepsilon_0 r^2} \cdot \bar{a}_r$$

Example 1.69 *Given* $D = \frac{3r}{r^2+1}\bar{a}_x$ *c/m² in spherical co-ordinate. Find th*

charge density.

☺**Solution:** $\nabla \cdot D = \rho_v$ (Volume charge density)

In spherical co-ordinate system,

$$\nabla \cdot D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$D = \frac{3r}{r^2+1}$$

$$\rho_v = \nabla \cdot D = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{3r}{r^2+1}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{3r^3}{r^2+1}\right)$$

$$= \frac{1}{r^2}\left[\frac{(r^2+1)\,9r^2 - 3r^3(2r)}{(r^2+1)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{9r^4+9r^2-6r^4}{(r^2+1)^2}\right]$$

$$= \frac{1}{r^2(r^2+1)^2}[3r^4+9r^2]$$

$$= \frac{1}{r^2(r^2+1)^2}\,3r^2[r^2+3]$$

$$= \frac{\mathbf{3(r^2+3)}}{\mathbf{(r^2+1)}}\ \mathbf{c/m^3}$$

☺

Example 1.70 *In spherical co-ordinate, let* $\rho = \frac{10^{-6}}{r}$ *c/m³, find the*

(a) *Total charge in the region described by* $10 \le r \le 20$ *m,* $28° \le \theta \le 31$ $0.9\pi \le \phi \le 0.96\,\pi$.

(b) *The total charge in a sphere of 1 mm radius centered at the origin.*

☺**Solution:**

(a) Total charge $Q = \iiint \rho\, dv$

$$\rho = \frac{10^{-6}}{r}, \quad dv = r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$Q = \int_{0.9\pi}^{0.96\pi}\int_{28}^{31}\int_{10}^{20}\frac{10^{-6}}{r}\,r^2\sin\theta\ dr\, d\theta\, d\phi$$

$$\mathbf{Q = 0.729\ \mu c}$$

(b) Total charge in a sphere at origin $Q = \iiint \rho\, dv$

$$= \int_0^R \frac{10^{-6}}{r}\,4\pi r^2 \cdot dr$$

$$= 2\pi R^2 \times 10^{-6}$$

But R = 1 mm $Q = 2\pi \times 10^{-6} \times 10^{-6}$

$$\mathbf{Q = 2\pi \times 10^{-12}\ Coulomb}$$

☺☺

Example 1.71 *Let* $D = 4x^2\bar{a}_x - 2y\bar{a}_y + 2z\bar{a}_z$. *Using Gauss's law, find he total charge enclosed by the region* $-a < x, y, z < a$.

☺**Solution:** According to the Gauss's law

$$\oint_s D \cdot ds = Q$$

Using divergence theorem,

$$\oint_s D \cdot ds = \iiint_v \nabla \cdot D\ dv = Q$$

$$\nabla \cdot D = \left[\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right](4x^2\bar{a}_x - 2y\bar{a}_y + 2z\bar{a}_z)$$

$$= 8x - 2 + 2 = 8x$$

$$Q = \iiint \nabla \cdot D \, dv = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} 8\,x\,dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} dy\,dz\,[\,4\,x^2\,]_{-a}^{a} = 0$$

$$Q = 0$$

The total charge enclosed by the volume is equal to zero. ☺

Example 1.72 *Find the charge in the volume defined by $0 \le x \le 1$ $0 \le y \le 1$ m and $0 \le z \le 1$ m if $\rho_v = 60\,xy^2$ c/m^3.*

☺Solution:

$$Q = \int dQ = \int_v \rho \cdot dv$$

$$= \int_0^1\int_0^1\int_0^1 60\,xy^2\;dx\;dy\;dz$$

$$= \int_0^1\int_0^1 [\,60\,xy^2\,z\,]_0^1\;dx\,dy$$

$$= \int_0^1\int_0^1 60\,xy^2\;dx\;dy$$

$$= \int_0^1 [\,20\,xy^3\,]_0^1\;dx$$

$$= \int_0^1 20\,x\;dx = 20\frac{x^2}{2}\Big|_0^1$$

$$= \mathbf{10\ c}$$

☺

Example 1.73 *A circular disc of radius 5 m with a volume charge dens $\rho_s = 10\sin\phi$ c/m^2 is enclosed by surface. What is the net flux crosses surface?*

☺Solution: $\psi = Q = \int dQ = \int \rho\;ds$

$$\psi = \int_0^{2\pi}\int_0^{5} (10\sin\phi)\;r\;dr\;d\phi$$

$$= \int_0^{2\pi} (10\sin\phi)\,\frac{r^2}{2}\,d\phi\,\Big|_0^5$$

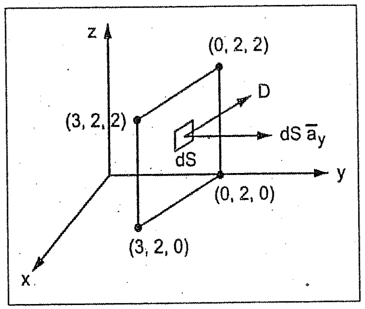
$$= \int_0^{2\pi} 5\;\sin\phi \cdot 25 \cdot d\phi$$

$$= \int_0^{2\pi} 125\;\sin\phi\;d\phi = -125\cos\phi\,\Big|_0^{2\pi}$$

$$= 0$$

The net fluxes crosses the surface is zero. ☺☺

Example 1.74 *What is electric flux, that passes the surface shown in Fig., if the displacement flux density is $D = y\,\bar{a}_x + x\,\bar{a}_y$ mc/m^2?*



☺Solution: $D = (y\,\bar{a}_x + x\,\bar{a}_y) \times 10^{-3}$ c/m^2

Differential area $ds = dx\,dz\,\bar{a}_y$

Differential flux $d\psi$ passing through ds is

$$d\psi = D \cdot ds$$

$$= (y\,\bar{a}_x + x\,\bar{a}_y) \times 10^{-3}\,dx\,dz\,\bar{a}_y$$

$$= x \times 10^{-3}\,dx\,dz$$

Total flux passing through the surface

$$\psi = \int_0^2 \int_0^3 x \times 10^{-3}\, dx\, dz = 10^{-3} \int_0^2 \frac{x^2}{2} \Big|_0^3 dz$$

$$= 10^{-3} \int_0^2 \frac{9}{2}\, dz = \mathbf{9 \times 10^{-3}\ c\ \ or\ \ 9\ mc}$$

TWO MARK QUESTIONS AND ANSWERS

1. ***Define vector and scalar field. Give an example.***

   A scalar field is specified by a single number at each point.

   E.g. Temperature and pressure of a gas.

   A vector field is specified by both magnitude and direction at each point i space.

   E.g. Velocity and acceleration of a field.

2. ***Define vector product of two vectors.***

   The vector product (cross product) of two vectors A and B is a vector whic is equal to the product of the magnitudes of A, B and sine of the angle betwee A and B.

$$A \times B = |A||B| \sin\theta$$

3. ***Find the dot product of the vectors A and B if $A = 2\,\bar{a}_x - 3\,\bar{a}_y + 4\,\bar{a}_z$ $B = -\bar{a}_x + 2\,\bar{a}_y + 2\,\bar{a}_z$.***

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z$$

$$= 2(-1) - 3(2) + 4(2) = 0$$

4. ***Given $A = 4\,\bar{a}_y + 8\,\bar{a}_z$ and $B = -2\,\bar{a}_y + 6\,\bar{a}_z$, find $A \cdot B$.***

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z$$

$$= 4(-2) + 8(6) = 40$$

5. ***Given two vectors: $\vec{P} = 3\,\vec{i} + 5\,\vec{j} + 2\,\vec{k}$ and $\vec{Q} = 2\,\vec{i} - 4\,\vec{j} + 3\,\vec{k}$. Determine the angular separation between them.***

$$\vec{P} = 3\,\vec{i} + 5\,\vec{j} + 2\,\vec{k}$$

$$\vec{Q} = 2\,\vec{i} - 4\,\vec{j} + 3\,\vec{k}$$

$$\cos\theta = \frac{\vec{P}\cdot\vec{Q}}{|\vec{P}||\vec{Q}|} = \frac{(3\,\vec{i} + 5\,\vec{j} + 2\,\vec{k})(2\,\vec{i} - 4\,\vec{j} + 3\,\vec{k})}{\sqrt{9+25+4}\ \sqrt{4+16+9}}$$

$$= \frac{6-20+6}{\sqrt{38}\ \sqrt{29}} = \frac{-8}{\sqrt{1102}}$$

$$\theta = \cos^{-1}(-0.241) = 103.94°$$

6. ***What are the conditions for two vectors $\bar{A}$ and $\bar{B}$ to be (a) parallel, (b) perpendicular.***

   (*a*) For parallel $\bar{A} \times \bar{B} = 0$ (cross product)

   (*b*) For perpendicular $\bar{A} \cdot \bar{B} = 0$ (dot product)

7. ***Show that the two vectors $\bar{A} = 6\,\bar{a}_x + \bar{a}_y - 5\,\bar{a}_z$ and $\bar{B} = 3\,(\bar{a}_x - \bar{a}_y + \bar{a}_z)$ are perpendicular to each other.***

$$\bar{A} \cdot \bar{B} = 6 \times 3 + 1 \times (-3) + (-5) \times (3) = 18 - 3 - 15 = 0$$

   ∴ Vectors $\bar{A}$ and $\bar{B}$ are perpendicular.

8. ***How can a vector field be expressed as the gradient of scalar field?***

   The gradient of any scalar field V is expressed as

$$\nabla V = \frac{\partial V}{\partial x}\,\bar{a}_x + \frac{\partial V}{\partial y}\,\bar{a}_y + \frac{\partial V}{\partial z}\,\bar{a}_z$$

   where V represents electric potential (scalar).

   Gradient of scalar is a vector. Potential gradient ∇V is a vector field.

9. ***State the condition for the vector $\overline{F}$ to irrotational.***

The vector F is said to be irrotational if $\nabla \times \overline{F} = 0$.

10. ***Show that the vector $H = 3y^4z^2\overline{a}_x + 4x^3z^2\overline{a}_y + 3x^2y^2\overline{a}_z$ is solenoidal.***

$$\nabla \cdot \overline{H} = \left(\overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y} + \overline{a}_z \frac{\partial}{\partial z}\right)(3y^4z^2\overline{a}_x + 4x^3z^2\overline{a}_y + 3x^2y^2\overline{a}_z)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(3x^2y^2)$$

$$= 0 + 0 + 0 = 0$$

$$\nabla \cdot \overline{H} = 0$$

Hence $\overline{H}$ is solenoidal.

11. ***What is the physical significance of divergence of $\overline{D}$?***

The divergence of the vector flux density $\overline{D}$ is the outflow from a sma closed surface per unit volume as the volume shrinks to zero.

$$\nabla \cdot \overline{D} = \underset{\Delta v \to 0}{\text{Lt}} \frac{\oint \overline{D} \cdot ds}{\Delta v} = \rho_v$$

This is equal to the volume charge density.

12. ***What is the physical significance of curl of a vector field?***

Curl indicates a measure of the tendency of a vector to rotate or twist.

Curl may be described as circulation per unit area.

The curl of electric field E must be zero, for the circulation is zer However, the curl of magnetic field is not zero.

13. ***Give practical examples for diverging and curling fields.***

Divergence indicates the spreading or diverging of quantity from a point.

Examples:

1. Leaking air from a ballon yields positive divergence
2. Rushing of air into drum under the carriage of a trair yields negati divergence.

Curl indicates a measure of the tendency of a vector to rotate or twist.

Example: When a leaf floats in sea water and its rotation is about z-axis, the curl of velocity is in z-direction.

14. ***What is del operator ∇? How is it used in curl, gradient and divergence?***

Del (∇) is a vector differential operator

$$\nabla = \overline{a}_x \frac{\partial}{\partial x} + \overline{a}_y \frac{\partial}{\partial y} + \overline{a}_z \frac{\partial}{\partial z}$$

$$\text{Curl H} = \begin{vmatrix} \overline{a}_x & \overline{a}_y & \overline{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\text{Gradient } \nabla = \nabla V = \overline{a}_x \frac{\partial V}{\partial x} + \overline{a}_y \frac{\partial V}{\partial y} + \overline{a}_z \frac{\partial V}{\partial z}$$

$$\text{Div B} = \nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

15. ***Express the divergence of a vector in the three system of orthogonal co-ordination.***

For rectangular co-ordinate system

$$\nabla \cdot B = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

For cylindrical co-ordinate system

$$\nabla \cdot B = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}$$

For spherical co-ordinate system

$$\nabla \cdot B = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, B_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, B_\theta) + \frac{\partial}{\partial \phi}(r\, B_\phi)\right]$$

16. Find the divergence of $\bar{F} = xy\,\bar{a}_x + yz\,\bar{a}_y + zx\,\bar{a}_z$.

$$\nabla \cdot \bar{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx)$$

$$\nabla \cdot \bar{F} = y + z + x$$

17. Find the divergence of $\bar{A}$ ***where*** $\bar{A} = e^{-y}(\cos x\ \bar{a}_x - \sin x\ \bar{a}_y)$.

$$\nabla \cdot \bar{A} = \frac{\partial}{\partial x}(e^{-y}\cos x) + \frac{\partial}{\partial y}(-e^{-y}\sin x)$$

$$= -e^{-y}\sin x + e^{-y}\sin x$$

$$= 0$$

18. Find the curl of $\bar{F} = x^2 y\,\bar{a}_x - y^2 z\,\bar{a}_y + z^2 x\,\bar{a}_z$.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -y^2 z & z^2 x \end{vmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial y}(z^2 x) - \frac{\partial}{\partial z}(-y^2 z)\right] - \bar{a}_y\left[\frac{\partial}{\partial x}(z^2 x) - \frac{\partial}{\partial z}(x^2 y)\right] + \bar{a}_z\left[\frac{\partial}{\partial x}(-y^2 z) - \frac{\partial}{\partial y}(x^2 y)\right]$$

$$= y^2\,\bar{a}_x - z^2\,\bar{a}_y - x^2\,\bar{a}_z$$

19. What is the divergence of curl of a vector?

$$\nabla \cdot \nabla \times H = 0$$

20. What is the value of $\nabla \times \nabla \times F$***?***

$$\nabla \times \nabla \times F = \nabla\,\nabla \cdot F - \nabla^2 F$$

21. Represent in unit vector along a vector $\bar{R} = 6\,\bar{a}_x + 8\,\bar{a}_y$.

$$\text{Unit vector}\quad \bar{a}_R = \frac{\bar{R}}{|\bar{R}|} = \frac{6\,\bar{a}_x + 8\,\bar{a}_y}{\sqrt{36+64}}$$

$$= 0.6\,\bar{a}_x + 0.8\,\bar{a}_y$$

22. Convert the rectangular co-ordinate (x, y, z) into spherical co-ordinates (r, θ, φ).

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}(z/r)$$

$$\phi = \tan^{-1}(y/x)$$

23. Represent point P (0, 1, 1) m given in cartesian co-ordinates, in spherical co-ordinates.

Cartesian co-ordinates $x = 0,\quad y = 1\text{ m},\quad z = 1\text{ m}.$

$$r^2 = x^2 + y^2 + z^2 = 0 + 1 + 1$$

$$r = \sqrt{2}\text{ m}$$

$$\cos\theta = \frac{z}{r} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

$$\tan\phi = \frac{y}{x} = \frac{1}{0} = \infty$$

$$\phi = 90^\circ$$

Spherical co-ordinates $r = \sqrt{2}\text{ m},\ \theta = 45^\circ,\ \phi = 90^\circ.$

24. Write down the expression for differential volume element in terms of spherical co-ordinates.

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

25. Express the value of differential volume in rectangular and cylindrical co-ordinate systems.

For rectangular co-ordinates: $dv = dx\ dy\ dz$

For cylindrical co-ordinates: $dv = \rho\, d\rho\, d\phi\, dz$

26. Write expressions for differential length in cylindrical and spherical co-ordinates.

For cylindrical co-ordinates

$$dl = \sqrt{(d\rho)^2 + (\rho\, d\phi)^2 + (dz)^2}$$

For spherical co-ordinates

$$dl = \sqrt{(dr)^2 + (r\, d\theta)^2 + (r \sin\theta\ d\phi)^2}$$

27. Define a unit vector and its value in cartesian co-ordinate axis.

Unit vector is having unit magnitude and directed along the co-ordinates axes.

For cartesian co-ordinates,

$$\bar{A} = \bar{a}_x A_x + \bar{a}_y A_y + \bar{a}_z A_z$$

where $\bar{a}_x$, $\bar{a}_y$ and $\bar{a}_z$ are unit vectors in the direction of x, y, z respectively.

$$\bar{r} = \bar{i}\, x + \bar{j}\, y + \bar{k}\, z$$

or

where $\bar{i}, \bar{j}$ and $\bar{k}$ are unit vectors.

28. Find the distance from A (1, 2, 3) to B (2, 0, –1) in rectangular co-ordinates.

$$\begin{aligned} r &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-1)^2 + (0-2)^2 + (-1-3)^2} \\ &= \sqrt{1+4+16} \\ &= \sqrt{21} \end{aligned}$$

29. Write the dot products of unit vectors in cylindrical and cartesian co-ordinate system.

For cartesian co-ordinate system,

$$\bar{a}_x \cdot \bar{a}_y = \bar{a}_y \cdot \bar{a}_z = \bar{a}_z \cdot \bar{a}_x = 0$$

For cylindrical co-ordinate system

$$\bar{a}_z \cdot \bar{a}_\rho = \bar{a}_z \cdot \bar{a}_\phi = 0$$

30. $\bar{a}_x \cdot \bar{a}_r = ?$, $\bar{a}_y \cdot \bar{a}_r = ?$

$$\bar{a}_x \cdot \bar{a}_r = \sin\theta \cos\phi$$

$$\bar{a}_y \cdot \bar{a}_r = \sin\theta \sin\phi$$

31. Transform a vector $\bar{A} = y\,\bar{I}_x - x\,\bar{I}_y + z\,\bar{I}_z$ into cylindrical co-ordinates.

$$\bar{A} = y\,\bar{I}_x - x\,\bar{I}_y + z\,\bar{I}_z$$

The rectangular co-ordinates are $(y, -x, z)$.

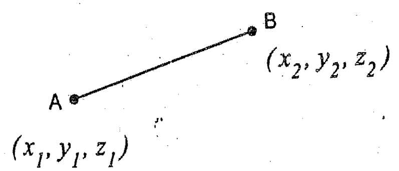
The cylindrical co-ordinates are (r, ϕ, z).

$$\begin{aligned} A_r &= A_x \cos\phi + A_y \sin\phi \qquad \text{where } \phi = \tan^{-1}(y/x) \\ &= y\cos\phi - x\sin\phi \\ \text{But } x &= r\cos\phi \text{ and } y = r\sin\phi \\ A_r &= r\sin\phi\cos\phi - r\cos\phi\sin\phi = 0 \\ A_\phi &= -A_x\sin\phi + A_y\cos\phi \\ &= -y\sin\phi - x\cos\phi \\ &= -[r\sin\phi \cdot \sin\phi + r\cos\phi\cos\phi] \\ &= -r\,[\sin^2\phi \cdot \cos^2\phi] \\ &= -r \\ A_z &= z \end{aligned}$$

Vector in cylindrical co-ordinates are

$$A = -r\,\bar{I}_\phi + z\,\bar{I}_z$$

32. Find the distance from A ($r = 5$, $\theta = 20°$, $\phi = 120°$) and B ($r = 2$, $\theta = 80°$, $\phi = 30°$) in spherical co-ordinates.



$$\begin{aligned} x_1 &= r\sin\theta\cos\phi \\ &= 5\sin 20° \cos 120° \\ &= -0.855 \\ y_1 &= r\sin\theta\sin\phi \end{aligned}$$

$$= 5 \sin 20° \sin 120° = 1.48$$

$$z_1 = r \cos\theta = 5 \cos 20°$$

$$= 4.7$$

$$x_2 = 2 \sin 80° \cos 30°$$

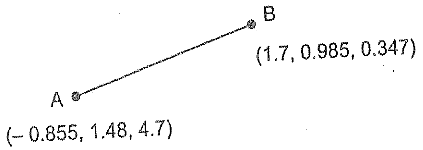
$$= 1.7$$

$$y_2 = 2 \sin 80° \sin 30°$$

$$= 0.985$$

$$z_2 = 2 \cos 80°$$

$$= 0.347$$



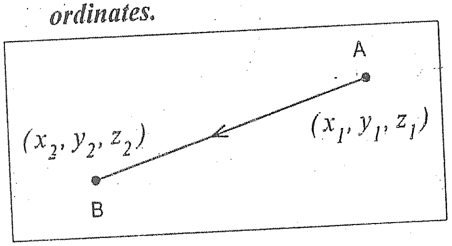
$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$= \sqrt{(1.7+0.855)^2 + (0.985-1.48)^2 + (0.347-4.7)^2}$$

$$= \sqrt{6.528 + 0.245 + 18.95}$$

$$d = 5.07$$

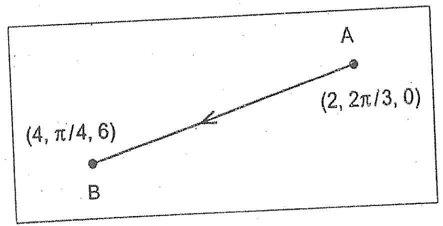
33. *Find the distance between (2, 2π/3, 0) and (4, π/4, 6) in cylindrical co-ordinates.*



$$x_1 = r \cos\phi = 2 \cos(2\pi/3) = -2$$

$$y_1 = r \sin\phi = 2 \sin(2\pi/3) = 1.732$$

$$z_1 = z = 0$$



$$x_2 = 4 \cos(\pi/4) = -2$$

$$y_2 = 4 \sin(\pi/4) = 2.83$$

$$z_2 = 6$$

$$d = \sqrt{(-2+2)^2 + (2.83-1.732)^2}$$

$$= \sqrt{(1.096)^2 + 36}$$

$$= 6.1$$

34. *Given point A(–2, 6, 3), find the spherical coordinate of point A.*

$$r = \sqrt{x^2+y^2+z^2}$$

$$= \sqrt{(-2)^2+6^2+3^2}$$

$$= \sqrt{4+36+9} = 7$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{3}{7}\right) = 64.62°$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$= \tan^{-1}\left(\frac{6}{-2}\right) = -71.56°$$

Spherical coordinates A (7, 64.62°, – 71.56°).

35. *Transform the vector $\bar{A} = y\,\bar{a}_x - x\,\bar{a}_y + z\,\bar{a}_z$ into cylindrical coordinates.*

Given: $\bar{A} = y\,\bar{a}_x - x\,\bar{a}_y + z\,\bar{a}_z$

$$(x, y, z) \rightarrow (y, -x, z)$$

Cylindrical coordinates:

$$r = \sqrt{x^2+y^2} = \sqrt{y^2+x^2}$$

$$\phi = \tan^{-1}(y/x)$$

$$= \tan^{-1}(-x/y)$$

$$z = z$$

36. *State divergence theorem.*

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field over a volume enclosed by the closed surface.

$$\oiint_s \mathrm{D} \cdot ds = \iiint_v \nabla \cdot \mathrm{D}\, dv$$

7. *State Stoke's theorem.*

The line integral of a vector around a closed path is equal to the surface integral of the normal component of its curl over any surface bounded by the path.

$$\oint \mathrm{H} \cdot dl = \iint \nabla \times \mathrm{H}\, ds$$

38. Define electric flux and flux density.

The lines of electric force is electric flux. It is denoted by χ.

$$\chi = Q\text{ (charge) coulomb}$$

Electric flux density is defined as electric flux per unit area. It is denoted by D.

$$D = \frac{Q}{A} \quad c/m^2$$

39. What is a point charge?

Point charge is one whose maximum dimension is very small in compariso with any other length.

40. What do you understand by linear, surface and volume charge densities.

Linear charge density: It is the charge per unit length (C/m) at a point o the line of charge.

$$\rho_l = \underset{\Delta l \to 0}{\mathrm{Lt}} \left(\frac{\Delta Q}{\Delta l}\right)$$

Surface charge density: It is the charge per surface area (C/m^2) at a poi on the surface of the charge

$$\rho_s = \underset{\Delta s \to 0}{\mathrm{Lt}} \left(\frac{\Delta Q}{\Delta s}\right)$$

Volume charge density: It is the charge per volume (C/m^3) at a point on th volume the charge.

$$\rho_v = \underset{\Delta v \to 0}{\mathrm{Lt}} \left(\frac{\Delta Q}{\Delta v}\right)$$

41. Define electric field intensity or electric field.

Electric field intensity is defined as the electric force per unit positi charge. It is denoted by E.

$$E = \frac{F}{Q} \quad N/c \text{ or } V/m$$

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

42. Name a few applications of Gauss's law in electrostatics.

Gauss's law is applied to determine the electric field intensity from a closed surface.

(*e.g.*) Electric field can be determined for charged shell, two concentric shell or cylinders, *etc.*

43. A uniform spherical volume charge distribution contains a total charge of 10^{-8} c. If the radius of this spherical volume is 2×10^{-2} m, find ρ_v.

$$\rho_v = \frac{Q}{\frac{4}{3}\pi r^3} = \frac{10^{-8}}{\frac{4}{3}\pi \times (2 \times 10^{-2})^3}$$

$$= \frac{10^{-2}}{\frac{4}{3}\pi \times 8} = 0.2984 \times 10^{-3} \ c/m^3$$

44. Why Gauss's law cannot be applied to determine the electric field due to finite line charge?

Gauss's law cannot be applied on non-Gaussian surface. It can be applied if the surface encloses the volume completely. Therefore Gauss's law cannot be applied to determine to electric field due to finite line charge.

45. A uniform surface charge of $\sigma = 2\ \mu\ c/m^2$ is situated at $z = 2$ plane. What is the value of flux density at P(1, 1, 1)m?

Given: $\sigma = 2\ \mu\ c/m^2$

Flux density due to surface charge of σ is given by

$$D = \frac{\rho_s}{2}\bar{a}_x = \frac{2 \times 10^{-6}}{2}\bar{a}_x$$

$$= 10^{-6}\,\bar{a}_x \ c/m^2$$

46. Electric flux density in a charge free region is given by $D = 10\,x\,\vec{a}_x + 5\,y\,\vec{a}_y + k\,z^2\,\vec{a}_z$ $\mu c/m^2$, find the constant k.

$$D = 10\,x\,\vec{a}_x + 5\,y\,\vec{a}_y + k\,z^2\,\vec{a}_z$$

$$\nabla \cdot D = 0$$

$$= \frac{\partial}{\partial x}(10x) + \frac{\partial}{\partial y}(5y) + \frac{\partial}{\partial z}(kz^2)$$

$$= 10 + 5 + 2kz = 0$$

$$2kz = -15$$

$$k = -7.5z$$

47. *Find the force of interaction between two charges spaced 10 cm apart in vacuum. The charges are 4 × 10^{-8} and 6 × 10^{-5} coulomb respectively. If th same charges are separated by the same distance in kerosine ($\varepsilon_r = 2$), what i the force of interaction?*

For vacuum $F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 r^2}$

$$= \frac{4 \times 10^{-8} \times 6 \times 10^{-5}}{4\pi \times \frac{1}{36\pi \times 10^9}} = 0.0216 \text{ N}$$

For kerosene $\varepsilon_r = 2$

$$F = \frac{Q_1 Q_2}{4\pi \varepsilon_0 \varepsilon_r r^2} = 0.0108 \text{ N}$$

48. *What is the relation between intensity of electric field $\bar{E}$ and electric flu density $\bar{D}$ in free space?*

$D = \varepsilon E$ c/m^2 where ε is permittivity of the medium

49. *Show by using Gauss's law: $\bar{D} = \varepsilon_0 \bar{E}$.*

The electric flux through any closed surface is

$$Q = \oint D \cdot ds \qquad \text{(Gauss's la}$$

For sphere, surface area $= 4\pi r^2$

$$Q = D \cdot 4\pi r^2$$

$$D = \frac{Q}{4\pi r^2}\bar{r}$$

But electric field intensity is

$$E = \frac{Q}{4\pi \varepsilon_0 r^2}\bar{r}$$

Substitute the value of D in the above equation

$$E = \frac{D}{\varepsilon_0}$$

$$\therefore D = \varepsilon_0 E$$

50. *State the principle of superposition of fields.*

The principle of superposition of fields is stated as follows.

The total electric field at a point is the algebraic sum of the individual electric field at that point.

For the three charges Q_1, Q_2 and Q_3 at distances r_1 r_2 and r_3 respectively, the total electric field at a point is given by

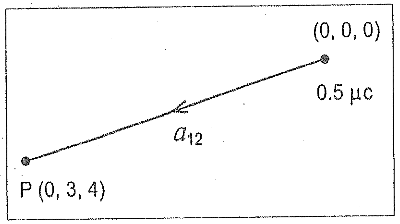
$$E = \frac{1}{4\pi\varepsilon}\left[\frac{Q_1}{r_1^2} + \frac{Q_2}{r_2^2} + \frac{Q_1}{r_3^2}\right] = \frac{1}{4\pi\varepsilon}\sum_{N=1}^{3}\frac{Q_N}{r_N^2}$$

51. *What is the electric field intensity at a distance of 20 cm from a charge of 2 μc in vacuum?*

$$E = \frac{Q}{4\pi \varepsilon_0 r^2} = \frac{2 \times 10^{-6}}{4\pi \times \frac{1}{36\pi \times 10^9} \times (0.2)^2}$$

$$= \frac{2 \times 9 \times 10^3}{0.04} = 450 \text{ kV/m}$$

52. *Find E at (0, 3, 4) m in cartesian co-ordinates due to point charge Q = 0.5 μc at the origin.*



$$\bar{a}_{12} = \frac{3\,\bar{a}_y + 4\,\bar{a}_z}{\sqrt{9+16}} = \frac{3\,\bar{a}_y + 4\,\bar{a}_z}{5}$$

$$E = \frac{Q}{4\pi\,\varepsilon\, r^2}\,\bar{a}_{12}$$

$$= \frac{0.5\times 10^{-6}}{4\pi \times \dfrac{1}{36\pi\times 10^9} \times (5)^2}\left(\frac{3\,\bar{a}_y + 4\,\bar{a}_z}{5}\right)$$

$$= \frac{9\times 0.5\times 10^3}{25}\left(\frac{3\,\bar{a}_y + 4\,\bar{a}_z}{5}\right)$$

$$= 108\ \bar{a}_y + 144\ \bar{a}_z$$

53. ***A uniform line charge, infinite in extent, with $\rho_l = 20$ nc/m lines along the Z-axis. Find E at (6, 8, 3) m.***

$$|r| = \sqrt{6^2+8^2+3^2} = \sqrt{36+64+9} = \sqrt{109}$$

$$E = \frac{\rho_l}{2\pi\,\varepsilon_0\, r} = \frac{20\times 10^{-9}}{2\pi \times \dfrac{1}{36\pi\times 10^9} \times \sqrt{109}}$$

$$= \frac{20\times 18}{\sqrt{109}} = 34.48\ \text{V/m}$$

54. ***A uniform line charge with $\rho_l = 5$ μc/m lies along the x-axis. Find $\bar{E}$ a***
(3, 2, 1) m.

$$E = \frac{\rho_l}{2\pi\,\varepsilon\, h}$$

$$h = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

$$E = \frac{5\times 10^{-6}}{2\pi \times \dfrac{10^{-9}}{36\pi} \times \sqrt{14}}$$

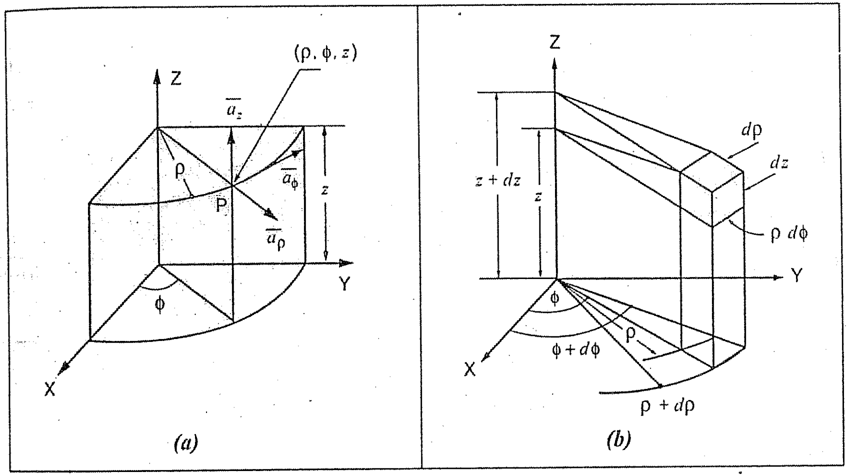
$$E = 24053.51\ \text{V/m}$$

55. ***What are the sources of electromagnetic fields?***

Sources of electromagnetic fields are

- *(i)* Stationary closed path in a time varying field.
- *(ii)* Time varying closed path in a static field.
- *(iii)* Time varying closed path in a time varying field.

56. ***Sketch a differential volume elements in cylindrical co-ordinates (r, φ, z) resulting from differential changes in three orthogonal co-ordinate directions.***



(a) *(b)*

57. ***State Coulomb's law.***

Coulomb stated that the force between two very small charged objects separated by a large distance compared to their size is proportional to the charge on each object and inversely proportional to the square of the distance between them.

$$F \propto \frac{Q_1\,Q_2}{r^2}$$

$$F = \frac{Q_1\,Q_2}{4\pi\,\varepsilon\, r^2}\,\bar{a}_{12}\ \text{Newtons}\qquad [\bar{a}_{12} = \text{Unit vector}]$$

58. ***State Gauss's law.***

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

$$\chi = Q$$

59. ***What are the sources of electric field and magnetic fields?***

Electric field is produced by electric charge.

Source: Presence of electric charge.

Magnetic field is produced by moving electric charge.

Source: Conduction current in a conductor.

60. ***What is the application of divergence theorem?***

The divergence theorem applies to time varying as well as static fields. This theorem is used to convert the volume integral of divergence of a vector into a closed surface integral.

61. ***What is the application of Stoke's theorem?***

The Stoke's theorem applies to time varying as well as static fields. This theorem is used to convert the surface integral of curl of a vector into a closed line integral.

SUMMARY

Any physical quantity can be represented either as a scalar or vector.

Scalar multiplication (dot product) of two vectors is scalar.

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z$$

Vector product $\bar{A} \times \bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$

$$\bar{A} \times \bar{B} = -\bar{B} \times \bar{A}$$

$$\bar{A} \cdot (\bar{B} \times \bar{C}) = (\bar{A} \times \bar{B}) \cdot \bar{C}$$

Condition for two vectors $\bar{A}$ and $\bar{B}$ to be parallel

$$\bar{A} \times \bar{B} = 0$$

Condition for two vectors A and B to be perpendicular

$$\bar{A} \cdot \bar{B} = 0$$

Gradient:

$$\text{Grad } V = \nabla V = \bar{a}_x \frac{\partial V}{\partial x} + \bar{a}_y \frac{\partial V}{\partial y} + \bar{a}_z \frac{\partial v}{\partial z}$$

Divergence:

$$\text{Div } \bar{A} = \nabla \cdot \bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Curl:

$$\text{Curl } A = \nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Vector $\bar{A}$ is ***solenoidal*** if $\nabla \cdot \bar{A} = 0$.

Vector $\bar{F}$ is ***irrotational*** if $\nabla \times \bar{F} = 0$.

Identities:

$$\nabla \cdot \nabla \times H = 0$$

$$\nabla \times \nabla V = 0$$

$$\nabla \times \nabla \times H = \nabla \nabla \cdot H - \nabla^2 H$$

For cartesian co-ordinate system (x, y, z)

Differential length $dl = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

Differential area $dx = dx\ dy = dy\ dz$

$= dz\ dx$

Differential volume $dv = dx\ dy\ dz$

For cylindrical co-ordinate system (ρ, ϕ, z)

The differential length $dl = \sqrt{(d\rho)^2 + (\rho\, d\phi)^2 + (dz)^2}$

The differential areas $ds = \rho \cdot d\rho\, d\phi$

$= d\rho\ dz$

$= \rho\ d\phi\ dz$

The differential volume $dv = \rho\ d\rho\ d\phi\ dz$

For spherical co-ordinate system (r, θ, ϕ)

The differential length $dl = \sqrt{dr^2 + (r\, d\theta)^2 + (r \sin\theta\, d\phi)^2}$

The differential areas $ds = r\, dr\ d\theta$

$= r \sin\theta\ d\phi\ dr$

$= r^2 \sin\theta\ d\theta\ d\phi$

The differential volume $dv = r^2 \sin\theta\ d\theta\ d\phi\ dr$

Transformation

Cartesian	Cylindrical	Spherical
x	$\rho = \sqrt{x^2+y^2+z^2}$	$r = \sqrt{x^2+y^2}$
y	$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$	$\phi = \tan^{-1}(y/x)$
z	$\phi = \tan^{-1}(y/x)$	z

Divergence theorem:

$$\iiint_v \nabla \cdot A\ dv = \oint\!\!\oint_s A \cdot ds$$

Stokes theorem:

$$\oint H \cdot dl = \iint_s \nabla \times H\ ds$$

Gauss's law:

$$\text{Flux} = \chi = \iint_s D \cdot ds = \iiint_v \rho_v\ dv = Q \text{ coulomb}$$

Coloumb's law:

$$\text{Force } F = \frac{Q_1 Q_2}{4\pi\, \varepsilon_0\, \varepsilon_r\, r^2}\ \bar{a}\ \text{ Newton}$$

Electric flux density:

$$D = \frac{Q}{4\pi r^2}\ \text{c/m}^2$$

$$D = \varepsilon E$$

Electric field intensity:

$$E = \frac{Q}{4\pi\, \varepsilon\, r^2}\ \text{V/m}$$

$$E = -\nabla V$$

Electric dipole moment: $m = Q\,d$ coulomb meter

Particulars	Electric field intensity (V/m)
At any point due to finite line charge	$E_x = \frac{\rho_l}{4\pi\, \varepsilon\, h}(\cos\alpha_1 + \cos\alpha_2)$ $E_y = \frac{\rho_l}{4\pi\, \varepsilon\, h}(\sin\alpha_2 - \sin\alpha_1)$
Point at bisector of finite line charge	$E = E_x = \frac{\rho_l}{2\pi\, \varepsilon\, h} \cos\alpha$
At any point due to infinite line charge	$E = E_x = \frac{\rho_l}{2\pi\, \varepsilon\, h}$

Particulars	Electric field intensity (V/m)
At any point due to charged circular disc	$E = \frac{\rho_s}{2\varepsilon}\left[1 - \frac{h}{\sqrt{h^2+R^2}}\right]$
At any point due to infinite plane charge sheet	$E = \frac{\rho_s}{2\varepsilon}$
At any point between two infinitely conducting planes	$E = \frac{\sigma}{\varepsilon}$
At any point inside the charged spherical shell	$E = 0$ $\quad r < a$
At any point outside the charged spherical shell	$E = \frac{Q}{4\pi\varepsilon r^2}$ $\quad r > a$
At any point between two shells	$E = \frac{Q_1+Q_2}{4\pi\varepsilon r^2}$ $\quad r > b > a$
Co-axial cylinders	$E = \frac{\rho_l}{2\pi\varepsilon r} = \frac{V}{r\, ln\left(\frac{b}{a}\right)}$

EXERCISE

1. Explain the following: Gradient, Divergence and Curl
2. Derive an expression for the curl of a vector using general curvilinea orthogonal co-ordinate system. Hence derive the expressions in terms o cartesian and spherical co-ordinate systems.
3. State and prove divergence theorem.
4. State and prove Stokes theorem.
5. Show that $A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$.
6. Show that the vector

   $\bar{H} = (y^2 - z^2 + 3yz - 2x)\,\bar{a}_x + (3xz + 2xy)\,\bar{a}_y + (3xy - 2xz + 2z)\,\bar{a}_z$ i

   both irrotational and solenoidal.

7. Show that the vector

   $\bar{E} = (3x^2 + 2y^2 + 1)\,\bar{a}_x + (4xy - 3y^2 z - 3)\,\bar{a}_y + (2 - y^3)\,\bar{a}_z$ is irrotational and find its scalar potential.
8. Find the value of 'a' so that the vector

   $\bar{D} = (x + 3y)\,\bar{a}_x + (y - 2z)\,\bar{a}_y + (x + az)\,\bar{a}_z$ is solenoidal. [*Ans.* $a = -2$]
9. Find the value of the constant a, b, c so that the vector

   $\bar{E} = (x + 2y + az)\,\bar{a}_z + (bx - 3y - z)\,\bar{a}_y + (4x + cy + 2z)\,\bar{a}_z$ is irrotational.

   [*Ans.* $a = 4,\ b = 2,\ c = 1$]
10. Using the Divergence theorem, evaluate

    $\iint \bar{E}\, ds = 4xz\,\bar{a}_x - y^2\,\bar{a}_y + yz\,\bar{a}_z$ over the cube bounded by

    $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1.$ [*Ans.* 3/2]
11. Show that the two vectors $A = 6\,\bar{a}_x + \bar{a}_y - 5\,\bar{a}_x$ and $B = 3\,(\bar{a}_x - \bar{a}_y + \bar{a}_z)$ are perpendicular to each other.
12. Give the cylindrical co-ordinates of the point whose cartesian co-ordinates are $x = 3, y = 4$ and $z = 5$. [*Ans.* $r = 5,\ \phi = 53.1°,\ z = 5$]
13. State and explain Coulomb's law.
14. Derive an expression for electric field due to an infinite long line charge.
15. State and prove Gauss's law.
16. Show that $E = -\nabla V$.
17. Find the electric field intensity due to the presence of co-axial cable with inner conductor of ρ_s c/m^2 and outer conductor of $-\rho_s$ c/m^2.
18. What is dipole? Derive the expression electric field intensity due to a dipole.
19. Derive the expressions for the electric field intensity at a point P which is situated 'h' metre away from the disc along its axis. The disc is charged uniformly with a charge density of ρ_s c/m^2.
20. A file straight line of charge of length 12 cm carries a uniformly distributed charge of 0.3 μc per cm length. Determine electric field intensity E at a point located at a distance of 3 cm above the wire and displaced 3 cm to right of, and beyond one end. [*Ans.* 5.215×10^6 V/m]

21. The charge of 0.2 μc is acted upon by a force by 0.1 N. Determine distance other charge of 0.45 μc. [*Ans.* 9 c

22. A charge of 2 μc placed in an electric field experiences a force of 0.08 What is the magnitude of electric field intensity? [*Ans.* 40 kV/

23. Two particles having charges 2×10^{-9} and 5×10^{-9} coulombs are space 0.8 m apart. Determine the electric field at a point A situated at a distance 0.5 m from each of the particle.

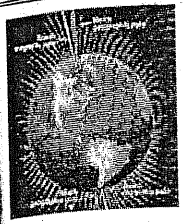
24. Point charges of 3×10^3 micro-micro coulombs are situated at each of the thre corners of a square whose side is 0.2 m. Find the magnitude and direction the electric field at the vacant corner.

25. Three point charges of 8×10^{-9} C, 32×10^{-9} C and 24×10^{-9} C are placed the corners A, B and C of a square ABCD having each side 4 cm. Find th electric intensity at the corner D. Assume the medium as air.

26. Four identical point charges Q coulombs each are placed at the four corners a square of side b. Find the force on a 1 C charge located at the centre of an one side.

27. Determine the magnitude of point charge which produces an electric fie intensity of 40 V/cm at a distance of 30 cm in vacuum.

28. Given $D = \frac{3r}{r^2+1}\,\bar{a}_r$ c/m³ in spherical co-ordinates, find the charge density.



2

ELECTROSTATICS-II

2.1. ELECTRIC POTENTIAL

Consider a uniform electric field E and a unit positive, say test charge q in the field E. If the test charge is moved from one point to another point in the electric field E, a force acts on the test charge due to the electric-field. The force is given by

$$F = q\,E$$

Since there is a movement of charge in the electric field from one point r_1 to another point r_2 against the direction of E, there will be a work done against the force.

$$W = -\int_{r_1}^{r_2} q\,E \cdot dr$$

Potential difference (V) is defined as the work done in moving a unit positive charge from one point to another in an electric field.

Work done on unit positive charge per charge is

$$V = \frac{W}{q}$$

$$V = -\int_{r_1}^{r_2} E \cdot dr \quad \text{Joules/Coulomb}$$

$$\text{But,} \quad E = \frac{Q}{4\pi\,\varepsilon\,r^2}$$

$$V = \frac{-Q}{4\pi\varepsilon}\int_{r_1}^{r_2}\frac{1}{r^2}\,dr = -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_{r_1}^{r_2}$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_2}-\frac{1}{r_1}\right] \text{ volts}$$

This is the potential difference between two points r_1 and r_2.

$$V = \frac{Q}{4\pi\varepsilon\, r_2} - \frac{Q}{4\pi\varepsilon\, r_1}$$

$$V = V_1 - V_2$$

If the test charge is moved from infinity to a given point ($r_1 = \infty$) in the electr

field,

$$V_2 = 0 \qquad \left[\because \frac{1}{r_1} = \right.$$

$$\text{then,} \quad V = V_1$$

Absolute potential or potential at a point is defined as the work done in movi a unit positive charge from infinity to a given point in an electric field.

$$V = \frac{Q}{4\pi\varepsilon\, r} \text{ volts}$$

Conservative Field

The potential difference, $V = -\int_{r_1}^{r_2} E\cdot dr$ is not independent of the path.

In a conservative field the work done in moving from one point to another independent of the path.

$$\oint E\cdot dr = 0$$

This shows no work is done in carrying the unit charge around any closed path.

Any field where the closed line integral of the field is zero, is said to be conservative field.

2.1.1. Relation between Electric Potential and Electric Field Intensity

If two points are separated by an infinitesimal distance dr, the work done by an external force in moving a unit positive charge from one point to the other will be

$$dw = dV = -E\cdot dr$$

Since the scalar potential V is a function of x, y, z the above equation can be written as

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -E\cdot dr$$

This can be rewritten as

$$\left[\bar{a}_x\frac{\partial V}{\partial x} + \bar{a}_y\frac{\partial V}{\partial y} + \bar{a}_z\frac{\partial V}{\partial z}\right]\cdot\left[\bar{a}_x dx + \bar{a}_y dy + \bar{a}_z dz\right] = -E\cdot dr$$

$$\nabla V\cdot dr = -E\cdot dr$$

$$E = -\nabla V$$

Thus the electric field strength at any point is just the negative of the potential gradient at that point. The negative sign shows that the direction of E is opposite to the direction in which V increases.

2.1.2. Electric Potential due to Uniformly Charged Infinite Line

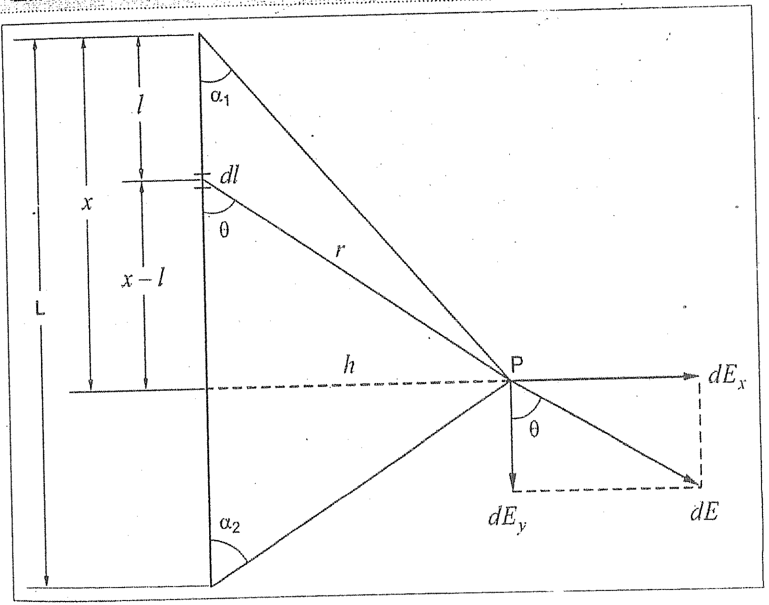


Fig. 2.1. Line charge

Consider a uniformly charged line of length 'L' whose linear charge density is ρ_l coulomb/metre. Consider a small element dl at a distance 'l' from one end of the charged line as shown in Fig.2.1. Let P be any point at a distance 'r' from the element dl.

The electric field at a point P due to the charge element $\rho_l\, dl$ is given by

$$dE = \frac{\rho_l\, dl}{4\pi \varepsilon r^2}$$

The x and y components of electric field dE are given by

$$dE_x = dE \sin\theta$$

$$dE_y = dE \cos\theta$$

$$\text{Then} \quad dE_x = \frac{\rho_l\, dl \sin\theta}{4\pi \varepsilon r^2}$$

From the Fig.,

$$x - l = h \cot\theta$$

$$-dl = -h \operatorname{cosec}^2\theta \cdot d\theta$$

$$\text{and} \quad \frac{h}{r} = \sin\theta$$

$$r = h \operatorname{cosec}\theta$$

$$dE_x = \frac{\rho_l \sin\theta\, d\theta}{4\pi \varepsilon h}$$

Substituting the values of dl and r in dE_x equation

$$dE_x = \frac{\rho_l (h \operatorname{cosec}^2\theta\, d\theta) \sin\theta}{4\pi \varepsilon (h \operatorname{cosec}\theta)^2}$$

The electric field E_x due to the entire length of line charge is given by

$$E_x = \int_{\alpha_1}^{\pi-\alpha_2} \frac{\rho_l \sin\theta\, d\theta}{4\pi \varepsilon h}$$

$$E_x = \frac{\rho_l}{4\pi \varepsilon h} \int_{\alpha_1}^{\pi-\alpha_2} \sin\theta\, d\theta$$

$$E_x = \frac{\rho_l}{4\pi \varepsilon h} \left[-\cos\theta\right]_{\alpha_1}^{\pi-\alpha_2}$$

$$E_x = \frac{\rho_l}{4\pi \varepsilon h} [\cos\alpha_1 + \cos\alpha_2]$$

Similarly for y component of E,

$$dE_y = \frac{\rho_l}{4\pi \varepsilon h} \cos\theta\, d\theta$$

$$E_y = \frac{\rho_l}{4\pi \varepsilon h} \int_{\alpha_1}^{\pi-\alpha_2} (\cos\theta)\, d\theta$$

$$E_y = \frac{\rho_l}{4\pi \varepsilon h} \left[\sin\theta\right]_{\alpha_1}^{\pi-\alpha_2}$$

$$E_y = \frac{\rho_l}{4\pi \varepsilon h} [\sin\alpha_2 - \sin\alpha_1]$$

Case 1: If the point P is at bisector of a line, then $\alpha_1 = \alpha_2 = \alpha$.

$$E_y = 0, \qquad \text{E becomes } E_x$$

$$E = \frac{\rho_l}{2\pi \varepsilon h}$$

Case 2: If the line is infinitely long, then $\alpha = 0$

$$E_y = 0, \qquad \text{E becomes } E_x$$

$$E = \frac{\rho_l}{2\pi \varepsilon h}$$

If test charge q is moved from one point r_1 to another point r_2 in the electric field, there will be work done against the force

$$W = -\int_{r_1}^{r_2} q\, E\, dh = -q \int_{r_1}^{r_2} \frac{\rho_l}{2\pi \varepsilon h}\, dh$$

$$= -\frac{q\, \rho_l}{2\pi \varepsilon} \int_{r_1}^{r_2} \frac{1}{h}\, dh = -\frac{q\, \rho_l}{2\pi \varepsilon}\, ln\, \frac{r_2}{r_1}$$

Work done per charge is potential

$$V = \frac{W}{q} = -\frac{\rho_l}{2\pi\varepsilon} \ln \frac{r_2}{r_1}$$

$$V = \frac{\rho_l}{2\pi\varepsilon} \ln \left[\frac{r_1}{r_2}\right]$$

2.1.3. Electric Potential due to Charged Circular Disc

Consider a circular disc of radius 'R' is charged uniformly with a charge density of ρ_s coulomb/m^2. Let P be any point on the axis of the disc at a distance from the centre. Consider an annular ring of radius 'r' and of radial thickness dr as shown in Fig.2.2. The area of the annular ring is $ds = 2\pi r\, dr$. The field intensity at point P due to the charged annular ring is given by

$$dE = \frac{\rho_s\, ds}{4\pi\varepsilon d^2}$$

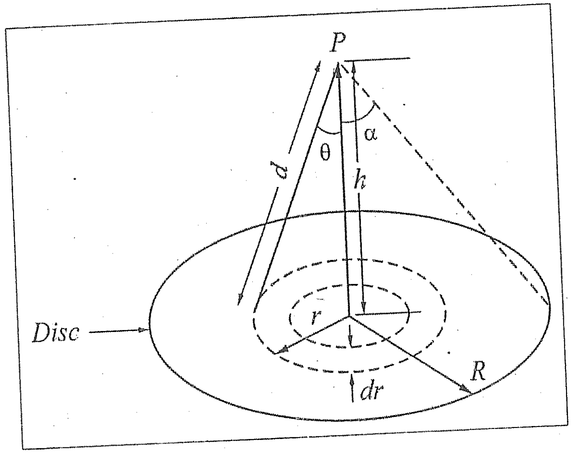


Fig. 2.2. Charged circular disc

Since the horizontal component of electric field intensity is zero, the vertica component is given by

$$dE_y = \frac{\rho_s \cdot ds}{4\pi\varepsilon d^2} \cos\theta$$

$$dE_y = \frac{\rho_s \cdot 2\pi r\, dr}{4\pi\varepsilon d^2} \cos\theta$$

From the Fig.,

$$r = h\tan\theta\,; \qquad dr = h\sec^2\theta \cdot d\theta$$

$$d = \frac{r}{\sin\theta}$$

Substituting the values of dr and d in dE_y equation

$$\text{then} \quad dE_y = \frac{\rho_s\, 2\pi r\, h\sec^2\theta \cdot d\theta \cos\theta \sin^2\theta}{4\pi\varepsilon r^2}$$

$$= \frac{\rho_s\, h\sec\theta \cdot \sin^2\theta\, d\theta}{2\varepsilon r}$$

$$= \frac{\rho_s \sec\theta \sin^2\theta\, d\theta}{2\varepsilon\tan\theta} \qquad \left[\because \frac{r}{h} = \tan\theta\right]$$

$$= \frac{\rho_s}{2\varepsilon}\sin\theta \cdot d\theta \qquad [\because \sec\theta\sin\theta = \tan\theta]$$

The total electric field due to the charged disc

$$E = \int_{\theta=0}^{\alpha} dE_y = \frac{\rho_s}{2\varepsilon}\int_0^{\alpha}\sin\theta \cdot d\theta$$

$$E = \frac{\rho_s}{2\varepsilon}(1-\cos\theta)$$

$$E = \frac{\rho_s}{2\varepsilon}\left[1 - \frac{h}{\sqrt{h^2+R^2}}\right]$$

Electric Potential

The electric potential 'V' at any point P due to the charged disc is

$$V = -\int_d^0 E \cdot dx = \int_d^0 \left[\frac{\rho_s}{2\varepsilon}(1-\cos\alpha)\right] dx$$

$$= -\frac{\rho_s}{2\varepsilon}(1-\cos\alpha)\int_d^0 dx = \frac{\rho_s}{2\varepsilon}(1-\cos\alpha)\, d$$

$$V = \frac{\rho_s}{2\,\varepsilon}\left(1 - \frac{h}{\sqrt{h^2+R^2}}\right)\sqrt{h^2+R^2}$$

$$V = \frac{\rho_s}{2\,\varepsilon}\left(\sqrt{h^2+R^2} - h\right) \text{ volts}$$

2.1.4. Electric Potential due to Electric Dipole

An electric dipole or simply dipole is that two equal and opposite charges separated by a very small distance. The product of charge and spacing is called ***electric dipole moment.***

Let Q and – Q be the two charges separated by a small distance d. The product of charge Q and spacing d is called dipole moment.

$$m = Q\,d$$

Let P be any point at distance of r_1, r_2 and r from +Q, –Q and mild point of dipole respectively as shown in Fig.2.3.

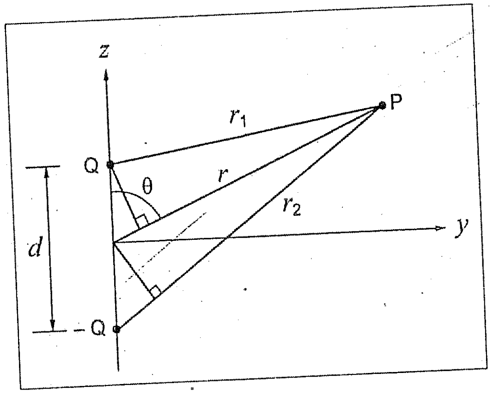


Fig. 2.3. Electric dipole

Potential at P due to +Q is $V_1 = \dfrac{Q}{4\pi\,\varepsilon\,r_1}$

Potential at P due to – Q is $V_2 = \dfrac{-Q}{4\pi\,\varepsilon\,r_2}$

The resultant potential at P is $V = V_1 + V_2 = \dfrac{Q}{4\pi\,\varepsilon}\left[\dfrac{1}{r_1} - \dfrac{1}{r_2}\right]$

If the point P is too far away from the dipole, the distances r_1 and r_2 are written as

$$r_1 = r - \frac{d}{2}\cos\theta$$

$$r_2 = r + \frac{d}{2}\cos\theta$$

The potential at P due to the dipole, .

$$V = \frac{Q}{4\pi\,\varepsilon}\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right)$$

$$= \frac{Q}{4\pi\,\varepsilon}\left(\frac{d\cos\theta}{\left(r - \frac{d}{2}\cos\theta\right)\left(r + \frac{d}{2}\cos\theta\right)}\right)$$

$$= \frac{Q}{4\pi\,\varepsilon}\left(\frac{d\cos\theta}{r^2 - \left(\frac{d}{2}\cos\theta\right)^2}\right)$$

$$= \frac{Q}{4\pi\,\varepsilon}\left(\frac{d\cos\theta}{r^2}\right) \qquad \left[\because \left(\frac{d}{2}\right) << r^2\right]$$

$$V = \frac{m\cos\theta}{4\pi\,\varepsilon\,r^2} \qquad [\because m = Q\,d]$$

This shows that the potential is directly proportional to the dipole moment and inversely proportional to the square of the distance.

2.1.5. Equipotential Surfaces

There are many points in an electric field at the same distance from the charge. All these points are having same electric potential because the electric potential is scalar quantity. If all the points which have equipotential are joining together to form a surface, then this surface is called as equipotential surface. Any surface over which the potential is constant in an electric field is called an equipotential surface. The equipotential surfaces are close to each other, where the electric field is strong. Electric field is perpendicular to the equipotential surface. The electric potential differences between any two points on the equipotential surface is zero. Therefore, work done in moving a charge over an equipotential surface is zero.

Consider a parallel plate capacitor with a potential across the plates. There is a constant electric field in the capacitor. All points on a plane parallel to the plates have same potential. The planes which are parallel to the plats are known as equipotential planes or surfaces as shown in Fig.2.4.

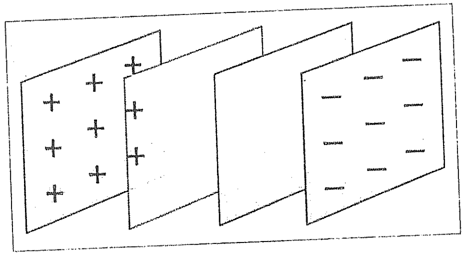


Fig. 2.4. Equipotential planes or surfaces

Consider a point charge Q at the origin. The electric potential at any distance from the origin is given by

$$V = \frac{Q}{4\pi \varepsilon_0 r}$$

All the points on the circumference of radius are having the same potential V The surface joining all these points is equipotential surface. At different radi different equipotential surfaces exist. The surfaces nearer to the point charge hav higher potential because potential is inversely proportional to the radial distanc (r).

Uniform field

A uniform field is one in which the electric field is constant at every point. common way to produce a uniform electric field is with a parallel plate capacito The electric potential increases linearly with distance. The uniform field is show in Fig.2.5.

Non-uniform field

A non-uniform field is one in which the electric field is not constant at eve point. In non-uniform field, the field lines tend to diverge in the direction. A no uniform field may be produced due to point charge as shown in Fig.2.6.

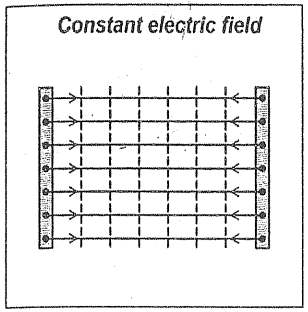


Fig. 2.5. Uniform field

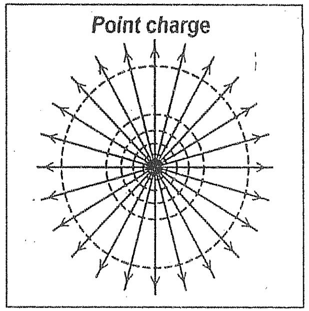


Fig. 2.6. Non-uniform field

Field utilization factor

Field utilization factor is defined as the ratio of mean electric field to the maximum electric field.

$$\eta = \frac{\text{Mean electric field}}{\text{Maximum electric field}}$$

2.2. CONDUCTORS AND DIELECTRICS

The electrical conduction properties of different elements and compounds can be explained in terms of the electrons having energies in the valence and conduction bands. A material may be classified as conductor, dielectric and semiconductor depending upon its energy structure.

If the valence band merges smoothly into a conduction band, then additional kinetic energy may be given to the valence electrons by an external field, resulting in an electron flow. The solid is called a metallic ***conductor.*** Since, there is no forbidden energy gap for metal, the required applied energy is extremely small. It has an excellent conductivity.

If the forbidden energy gap between valence band and conduction band of a material is high, it requires large applied energy to conduct. The material is called a ***dielectric.*** In dielectric, electrons are tightly bounded.

In electromagnetics, conductors and dielectrics are defined based on the ratio of conduction to displacement currents. Ratio of conduction current to displacement current is $\sigma/\omega\varepsilon$ where σ is the conductivity of the medium (material), ε is permittivity of the medium and ω is angular frequency. $\sigma/\omega\varepsilon = 1$ is considered to mark the dividing line between conductors and dielectrics. For good conductors $\sigma/\omega\varepsilon$ is much greater than 1 and for good dielectric $\sigma/\omega\varepsilon$ is much less than 1 in the radio frequency range.

The molecules of dielectric material may be of two types. They are non-polar molecules and polar molecules. The molecules whose centre of positive and negative charges coincide are called non-polar molecules. The molecules whose centre of charges are displaced from each other are called polar molecules. When dielectric material made up of non-polar molecules is placed in an electric field, a displacement of negative and positive charges takes place in opposite directions and produce a dipole which is aligned with the electric field.

2.2.1. Dielectric Polarization

Dipole may be described by its dipole moment P. If Q is the charge and l is the vector (distance) from the negative to the positive charge, the dipole moment is given by

$$P = Q \cdot l$$

If there are n dipole per unit volume ΔV, then there are $n\,\Delta V$ dipoles, and the total dipole moment is given by

$$P_{total} = \sum_{i=1}^{n\Delta V} P_i$$

When electric field is applied across the dielectric, dipoles are formed in a particular direction due to separation of charges in the material, the dielectric is said to be polarized.

Polarization is defined as dipole moment per unit volume.

$$P = \underset{\Delta V \to 0}{\text{Lt}} \frac{1}{\Delta V} \sum_{i=1}^{n\Delta V} P_i \quad \text{C/m}^2$$

If the dipoles are aligned in random orientation, polarization P has zero value, whereas if dipoles are aligned in same direction, polarization P has a significant value.

Consider a bound charge Q_b across a small element surface ds in dielectric containing non-polar molecules. Then

$$Q_b = -\oint_s P \cdot ds$$

If Q is the free charge enclosed by the surface s, the total enclosed charge is

$$Q_T = Q_b + Q$$

By Gauss's law, $\quad Q_T = \oint_s \varepsilon_0\, E \cdot ds$

The free charge enclosed,

$$Q = Q_T - Q_b$$

$$= \oint_s \varepsilon_0\, E \cdot ds + \oint_s P \cdot ds = \oint_s (\varepsilon_0 E + P)\, ds$$

But $\quad Q = \oint_s D \cdot ds$

Equating these two equations,

$$\therefore \quad D = \varepsilon_0 E + P$$

Polarization P can be written as

$$P = \chi\, \varepsilon_0 E$$

where, χ is electric susceptibility.

$$\therefore \quad D = \varepsilon_0 E + \chi\, \varepsilon_0\, E = \varepsilon_0\, E\,(1+\chi) = \varepsilon_0\, \varepsilon_R\, E$$

where, $\quad \varepsilon_R = \chi + 1$

2.3. POISSON'S AND LAPLACE'S EQUATIONS

According to Gauss's law in point form, the divergence of electric flux density is equal to the volume charge density.

$$\nabla \cdot D = \rho_v$$

But $\quad D = \varepsilon E$

$$\nabla \cdot (\varepsilon E) = \rho_v$$

$$\varepsilon\, \nabla \cdot E = \rho_v$$

$$\nabla \cdot E = \frac{\rho_v}{\varepsilon}$$

But $\quad E = -\nabla V$

$$\nabla \cdot (-\nabla V) = \frac{\rho_v}{\varepsilon}$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

This is the ***Poisson's equation.***

For cartesian co-ordinate system

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Poisson's equation for cartesian co-ordinate system is written as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

For cylindrical co-ordinate system, the Poisson's equation is

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$

For spherical co-ordinate system, the Poisson's equation is

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$= -\frac{\rho_v}{\varepsilon}$$

If the volume charge density (ρ_v) is zero, then

$$\nabla^2 V = 0$$

This is ***Laplace's equation.*** The operator ∇^2 is called Laplacian operator.

2.4. CAPACITANCE

2.4.1. Concept of Capacitance

A capacitor is an electric device which consists of two conductors separated by dielectric medium.

Consider a capacitor composed of two conducting plates of area 'A' separated by a dielectric medium whose permittivity is 'ε'. The space separation between the plates is '*d*'. If potential 'V' is applied across the plates, the positive charge Q is deposited on one plate and the negative charge – Q is deposited on other plate as shown in Fig.2.7. The net charge is zero.

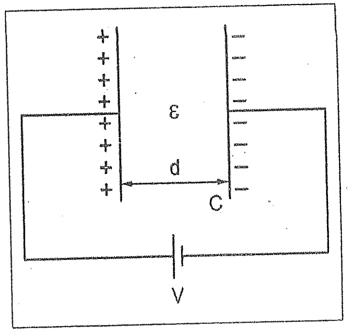


Fig. 2.7. Capacitor

The capacitance of two conducting planes is defined as the ratio of magnitude of charge on either of the conductor to the potential difference between conductors. It is given by

$$C = \frac{Q}{V}$$

The unit of capacitance is coulombs/volt or Farad.

Assume that there is a uniform charge density D over the plates and dielectric medium.

$$D = \frac{Q}{A} \quad c/m^2$$

It is also written in terms of electric field E as

$$D = \varepsilon E$$

$$\frac{Q}{A} = \varepsilon E$$

$$Q = A \varepsilon E$$

But electric field is given by

$$E = \frac{V}{d} \quad V/m$$

Substituting the value of E in the above equation

$$Q = A \varepsilon \frac{V}{d}$$

$$\frac{Q}{V} = \frac{A \varepsilon}{d}$$

Capacitance is given by $$C = \frac{Q}{V}$$

$$\therefore \quad C = \frac{A\,\varepsilon}{d}$$

$$C = \frac{A\,\varepsilon_0\,\varepsilon_r}{d} \quad \text{Farad}$$

$$\text{where,} \quad \varepsilon = \varepsilon_0\,\varepsilon_r$$

2.4.2. Capacitance of Two Dielectric Media

Consider a parallel plate capacitor consists of two dielectrics as shown in Fig.2.8.

The relative permittivity of dielectric medium 1 and medium 2 are ε_{r_1} and ε_{r_2} respectively. If the potential across the capacitor is V, the potential difference across medium 1 and medium 2 are V_1 and V_2 respectively.

$$V = V_1 + V_2$$

Let E_1 and E_2 be the field intensities in the medium 1 and medium 2 respectively. Then,

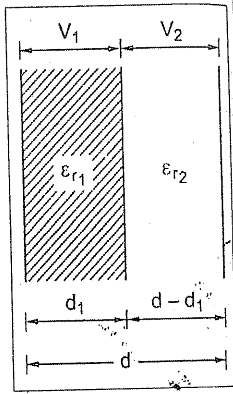


Fig. 2.8. Capacitor with two dielectrics

$$V_1 = E_1\,d_1$$

$$V_2 = E_2\,(d - d_1)$$

$$\therefore \quad V = V_1 + V_2$$

$$= E_1\,d_1 + E_2\,(d - d_1)$$

The electric flux density $D = \frac{Q}{A}$ will be the same in both the medium. Th electric field intensities for both the medium are given by

$$E_1 = \frac{D}{\varepsilon_0\,\varepsilon_{r_1}} = \frac{Q}{A\,\varepsilon_{r_1}\,\varepsilon_0}$$

$$E_2 = \frac{D}{\varepsilon_0\,\varepsilon_{r_2}} = \frac{Q}{A\,\varepsilon_{r_2}\,\varepsilon_0}$$

The applied potential $V = E_1\,d_1 + E_2\,(d - d_1)$

$$V = \frac{Q}{A\,\varepsilon_0}\left[\frac{d_1}{\varepsilon_{r_1}} + \frac{d - d_1}{\varepsilon_{r_2}}\right]$$

$$\frac{Q}{V} = \frac{A\,\varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{d - d_1}{\varepsilon_{r_2}}}$$

$$C = \frac{A\,\varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{d - d_1}{\varepsilon_{r_2}}}$$

The capacitance $C = \frac{A\,\varepsilon_0\,\varepsilon_{r_1}\,\varepsilon_{r_2}}{d_1\,\varepsilon_{r_2} + (d - d_1)\,\varepsilon_{r_1}}$

If medium 1 is air, $\varepsilon_{r_1} = 1$ and for medium 2, $\varepsilon_{r_2} = \varepsilon_r$.

The capacitance of capacitor is

$$C = \frac{A\,\varepsilon_0\,\varepsilon_r}{d_1\,\varepsilon_r + (d - d_1)}$$

2.4.3. Capacitance of Three Dielectric Media

Consider a parallel plate capacitor consists of three dielectrics as shown in Fig.2.9. Let ε_{r_1}, ε_{r_2} and ε_{r_3} be the relative permittivity and thickness d_1, d_2 and d_3 of the medium 1, medium 2 and medium 3 respectively.

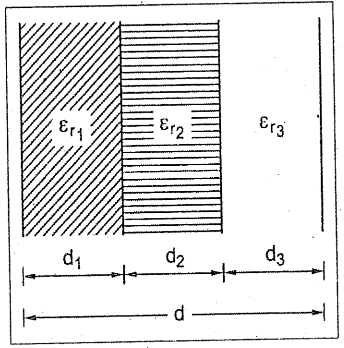


Fig. 2.9. Capacitor with three dielectrics

If E_1, E_2 and E_3 are the electric field intensity of the medium 1, 2, and 3 respectively, the corresponding applied potential is $V = V_1 + V_2 + V_3$.

$$V = E_1 d_1 + E_2 d_2 + E_3 d_3$$

Since the flux density $D = \frac{Q}{A}$ is same in the three medium, the electric field intensity is,

$$E_1 = \frac{D}{\varepsilon_0 \varepsilon_{r_1}} = \frac{Q}{A \varepsilon_0 \varepsilon_{r_1}}$$

$$E_2 = \frac{Q}{A \varepsilon_0 \varepsilon_{r_2}}$$

$$E_3 = \frac{Q}{A \varepsilon_0 \varepsilon_{r_3}}$$

$$\text{Then,} \quad V = \frac{Q}{A \varepsilon_0 \varepsilon_{r_1}} d_1 + \frac{Q}{A \varepsilon_0 \varepsilon_{r_2}} d_2 + \frac{Q}{A \varepsilon_0 \varepsilon_{r_3}} d_3$$

$$V = \frac{Q}{A \varepsilon_0} \left[\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}} + \frac{d_3}{\varepsilon_{r_3}} \right]$$

$$\frac{Q}{V} = \frac{A \varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}} + \frac{d_3}{\varepsilon_{r_3}}}$$

The capacitance is given by $$C = \frac{A \varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}} + \frac{d_3}{\varepsilon_{r_3}}}$$

Capacitance of an Isolated Sphere

Consider a sphere of radius r having a charge Q coulombs as shown in Fig.2.10.

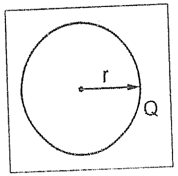


Fig. 2.10. Sphere of charge Q

The potential is the work done per unit charge in carrying a positive test charge from infinity to the sphere. The absolute potential is given by

$$V = -\int_{\infty}^{r} E \cdot dr$$

$$\text{But} \quad E = \frac{Q}{4\pi \varepsilon r^2} = \frac{Q}{4\pi \varepsilon} \int \frac{dr}{r^2}$$

$$V = \frac{Q}{4\pi \varepsilon r}$$

The capacitance of isolated sphere is

$$C = \frac{Q}{V} = 4\pi \varepsilon r$$

$$C = 4\pi \varepsilon r \quad \text{Farads} \quad \text{where } \varepsilon = \varepsilon_0 \varepsilon_r$$

If the medium is air, then C becomes

$$C = 4\pi \varepsilon_0 r \quad \text{Farad}$$

2.4.4. Capacitance of Concentric Spheres

Consider two concentric spheres of inner radius of 'a' and outer radius 'b'. Let ε_r be the permittivity of dielectric medium between the inner and outer spheres as shown in Fig.2.11.

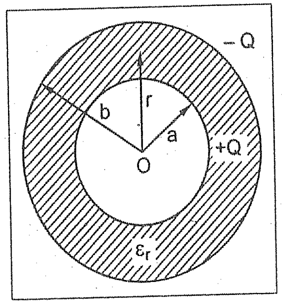


Fig. 2.11. Capacitance of concentric spheres

If the charge Q is distributed uniformly over the outer surface of the inner sphere, there will be equal and opposite charge induced on their outer surface of the outer sphere.

The electric field intensity at any point in between inner sphere and outer sphere $(a \le r \le b)$ is given by

$$E = \frac{Q}{4\pi \varepsilon r^2} \quad (a \le r \le b)$$

The potential difference between the sphere is

$$V = -\int_b^a \frac{Q}{4\pi \varepsilon r^2} \cdot dr = -\frac{Q}{4\pi \varepsilon} \int_b^a \frac{dr}{r^2}$$

$$= \frac{Q}{4\pi \varepsilon} \left[\frac{1}{r}\right]_b^a$$

$$= \frac{Q}{4\pi \varepsilon} \left[\frac{1}{a} - \frac{1}{b}\right]$$

$$= \frac{Q}{4\pi \varepsilon} \left[\frac{b-a}{ab}\right]$$

The capacitance of two concentric sphere is

$$C = \frac{Q}{V} = 4\pi \varepsilon \left[\frac{a\,b}{b-a}\right]$$

$$C = 4\pi \varepsilon \left[\frac{a\,b}{b-a}\right]$$

2.4.5. Capacitance of Co-axial Cables (Cylinders)

Consider a coaxial cable of inner radius '*a*' and outer radius '*b*' as shown in Fig.2.12. The relative permittivity of dielectric filled in between two coaxial cylinders is ε_r. A potential difference V is applied in between two cylinders. The two cylinders are charged at the rate of ρ_l c/m. It is assumed that inner cylinder has charge of ρ_l c/m and outer cylinder has charge of $-\rho_l$ c/m. By applying Gauss's law, the electric field E at any distance *r* from the axis of cylinder is given by

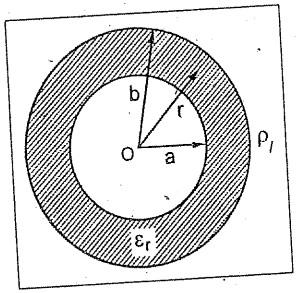


Fig. 2.12. Capacitance of coaxial cable

$$E = \frac{\rho_l}{2\pi \varepsilon r}$$

The potential difference between two coaxial cable is

$$V = -\int_b^a E\, dr$$

$$= -\frac{\rho_l}{2\pi \varepsilon} \int_b^a \frac{dr}{r}$$

$$V = \frac{\rho_l}{2\pi \varepsilon}\, ln\left[\frac{b}{a}\right]$$

The capacitance of coaxial cable / unit length is

$$C = \frac{\rho_l}{V} = \frac{2\pi \varepsilon}{ln\,(b/a)} \quad \text{F/m}$$

$$C = \frac{2\pi \varepsilon_0 \varepsilon_r}{ln\,(b/a)} \quad \text{F/m} \qquad [\because\ \varepsilon = \varepsilon_0 \varepsilon_r]$$

2.4.6. Capacitance of Parallel Conductors (Transmission Lines)

Consider a two parallel conductor of radius '*a*' separated by a distance '*d*' as shown in Fig.2.13. If the conductor A has charge of ρ_l c/m along its length, this will induce charge of $-\rho_l$ c/m on the conductor B.

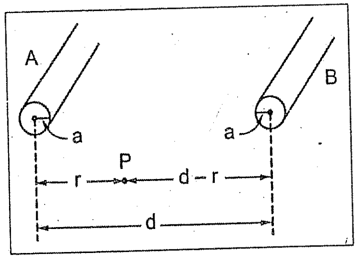


Fig. 2.13. Transmission lines (Parallel wires)

The electric field intensity at any point P with a distance *r* from the conductor A is algebraic sum of electric field intensity at P due to conductor A and conductor B.

$$E = \frac{\rho_l}{2\pi \varepsilon r} + \frac{\rho_l}{2\pi \varepsilon (d-r)} = \frac{\rho_l}{2\pi \varepsilon}\left(\frac{1}{r} + \frac{1}{d-r}\right)$$

The potential difference between the conductors is given by

$$V = -\int E \cdot dr$$

$$= -\frac{\rho_l}{2\pi \varepsilon}\int_{d-a}^{a}\left(\frac{1}{r} + \frac{1}{d-r}\right) dr$$

$$= -\frac{\rho_l}{2\pi \varepsilon}\left(ln\,\frac{a}{d-a} + ln\,\frac{a}{d-a}\right)$$

$$= -\frac{\rho_l}{2\pi \varepsilon}\left(2\,ln\,\frac{a}{d-a}\right)$$

$$V = \frac{\rho_l}{\pi \varepsilon}\,ln\left(\frac{d-a}{a}\right)$$

The capacitance per unit length between two parallel conductors

$$C = \frac{\rho_l}{V} = \frac{\pi \varepsilon}{ln\left(\frac{d-a}{a}\right)} \quad \text{F/m}$$

$$\text{If} \quad d >> a, \quad C = \frac{\pi \varepsilon}{ln\,\frac{d}{a}} \quad \text{F/m}$$

If the dielectric medium between two conductor is air (for transmission lines) $\varepsilon_r = 1$. *i.e.,* $\varepsilon = \varepsilon_0$.

$$C = \frac{\pi \varepsilon_0}{ln\left(\frac{d}{a}\right)} \quad \text{F/m}$$

2.4.7. Serial and Parallel Combination of Capacitors

Two capacitors are connected in series as shown in Fig.2.14. If potential difference V is applied across the two capacitors, V_1 and V_2 are the potential differences of capacitors C_1 and C_2 respectively.

$$V = V_1 + V_2$$

Since the charge acquired by each capacitor is same, then

$$V = \frac{Q}{C_1} + \frac{Q}{C_2}$$

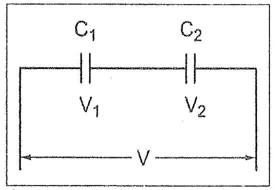


Fig. 2.14. Capacitors in series

The equivalent capacitance C is given by

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

If N number of capacitors are connected in series, the equivalent capacitance is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots\ldots \frac{1}{C_N}$$

$$\frac{1}{C} = \sum_{n=1}^{N} \frac{1}{C_n}$$

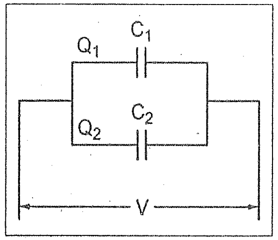


Fig. 2.15. Capacitors in parallel

The two capacitors of value C_1 and C_2 are connected in parallel as shown in Fig.2.15. The voltage source V is applied to this combination. The total charge acquired by the equivalent capacitor is the sum of the charges on individual capacitors.

$$Q = Q_1 + Q_2$$

$$= C_1 V + C_2 V$$

The equivalent capacitor C is given by

$$C V = C_1 V + C_2 V$$

$$C = C_1 + C_2$$

If N number of capacitors are connected in parallel, the equivalent capacitor is given by

$$C = C_1 + C_2 + C_3 + \ldots\ldots + C_N$$

$$C = \sum_{n=1}^{N} C_n$$

2.5. ELECTROSTATIC ENERGY AND ENERGY DENSITY

The capacitor stores the electrostatic energy equal to work done to build up the charge. If a voltage source is connected across the capacitor, the capacitor charges. Potential is defined as the work done per unit charge.

$$V = \frac{dW}{dQ}$$

The work done $dW = V \cdot dQ$

$$\text{But } V = \frac{Q}{C}$$

The capacitor is charged to the value of Q. The total work done is

$$W = \int_0^Q \frac{Q}{C}\, dQ = \frac{1}{C}\left[\frac{Q^2}{2}\right]_0^Q$$

$$W = \frac{Q^2}{2C} \text{ Joules}$$

But $\quad Q = CV$

$$W = \frac{1}{2} C V^2 \text{ Joules}$$

(or) $\quad W = \frac{1}{2} Q V \text{ Joules}$

Energy Density

Consider a elementary cube of side Δd parallel to the plates of a capacitor as shown in Fig.2.16.

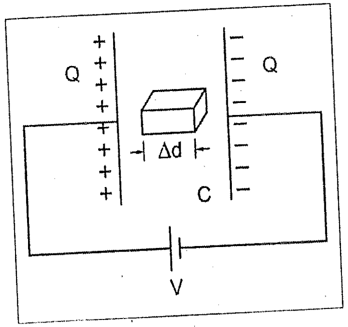


Fig. 2.16. Energy storage in capacitor

The capacitance of elemental capacitor is

$$\Delta C = \frac{\varepsilon A}{\Delta d} = \frac{\varepsilon (\Delta d)^2}{\Delta d} = \varepsilon \Delta d$$

Energy stored in the elemental capacitor is

$$\Delta W = \frac{1}{2} \Delta C \, (\Delta V)^2$$

But potential difference across the elementary cube is

$$\Delta V = E \cdot \Delta d$$

where E is the electric field exist in the cube.

Substituting the values of ΔC and ΔV in energy stored equation.

The stored energy $\quad \Delta W = \frac{1}{2} (\varepsilon \cdot \Delta d)\, (E \cdot \Delta d)^2 = \frac{1}{2} \varepsilon E^2 (\Delta d)^3$

$$= \frac{1}{2} \varepsilon E^2 \Delta v$$

where, $\Delta v = (\Delta d)^3$ is elementary volume.

The energy density is given by

$$\frac{\Delta W}{\Delta v} = \frac{1}{2} \varepsilon E^2 \qquad [\because D = \varepsilon E]$$

$$\frac{\Delta W}{\Delta v} = \frac{1}{2} D \cdot E \qquad \text{Joules/m}^3$$

2.6. BOUNDARY CONDITIONS FOR ELECTRIC FIELD

The solution of Poisson's or Laplace equation should always satisfy certain conditions at the boundaries between different media in the field. These conditions pertaining to normal component of the displacement, tangential component of the electric field intensity and refraction lines of forces are called boundary conditions. The statements of boundary conditions are:

(a) ***The tangential component of electric field E is continuous at the surface. That is E is the same just outside the surface as it is just inside the surface.***

(b) ***The normal component of electric flux density is continuous if there is no surface charge density. Otherwise D is discontinuous by an amount equal to the surface charge density.***

(*a*) Consider a interface or boundary between two dielectrics of dielectric constants ε_1 and ε_2 in an electric field.

Fig.2.17 shows the boundary surface between two dielectric media. Consider a rectangle of length Δy and width Δx at the boundary of two dielectric media as shown in Fig.2.17. In an electrostatic field, the voltage around any closed path must be zero.

$$V = \oint E \cdot dl = 0$$

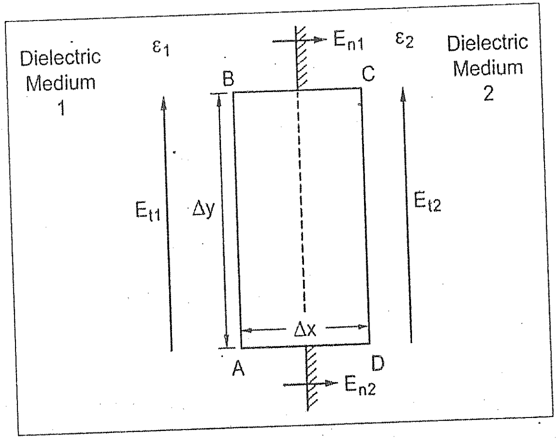


Fig. 2.17. Boundary surface between two dielectric media

Apply this to the rectangular path ABCD, in which AB is just inside medium 1 and CD is just inside medium 2, as shown in Fig.2.17.

$$\oint E \cdot dl = E_{t1}\,\Delta y + E_{n1}\,\Delta x - E_{t2}\,\Delta y - E_{n2}\,\Delta x$$

where, E_{t1} and E_{t2} are the average tangential components of E along paths AB and CD and E_{n1} and E_{n2} are the average normal components of E along the paths BC and AD. As the sides AB and CD are brought closer together, the lengths BC and AD approach zero *i.e.*, $\Delta x \to 0$.

Then, $$E_{t1}\,\Delta y - E_{t2}\,\Delta y = \oint E \cdot dl = 0$$

$$E_{t1} = E_{t2}$$

The tangential component of E is continuous at the boundary.

(*b*) Consider a pill box of surface area ds at the boundary of two dielectrics of dielectric constants ε_1 and ε_2 in an electric field, and it is assumed that there are no free charges on the boundary surface as shown in Fig.2.18.

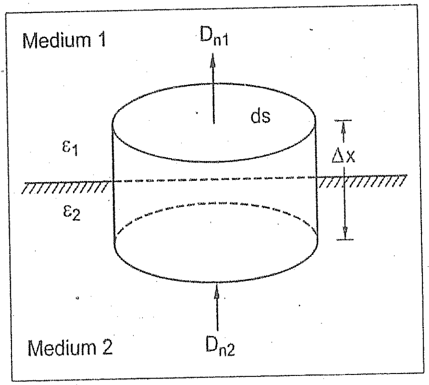


Fig. 2.18. A bill box at the boundary surface

Apply Gauss's law to the pill box at the boundary. Since there are no charges enclosed by the pill box, the surface integral of electric flux density over the pill box surface is zero.

$$\int_s D_{n1}\, ds - D_{n2}\, ds = 0$$

where, D_{n1} is electric flux density in medium 1.

D_{n2} is electric flux density in medium 2.

ds is the surface area of pill box.

$$D_{n1} = D_{n2}$$

The normal components of electric displacement or flux density are continuous across the boundary between two media *i.e.*, they are equal.

If the charges are enclosed by the pill box, the surface integral of electric flux displacement (flux density) over the pill box surface is

$$\int_s D_{n1}\, ds - \int_s D_{n2}\, ds = \int \rho_v\, dv$$

$$= \int \rho_v\, \Delta x\, ds \qquad \text{[Gauss's law]}$$

$$D_{n1}\, ds - D_{n2}\, ds = \rho_v\, \Delta x\, ds$$

where, ρ_v is the volume charge density enclosed by the pill box.

Δx is the height of the pill box.

If the height Δx is reduced to zero across the boundary $\Delta x \to 0$.

$$\underset{\Delta x \to 0}{\text{Lt}} \; \rho_v \, \Delta x = \rho_S \quad \text{(surface charge density)}$$

Then, $$D_{n1} - D_{n2} = \rho_S$$

The normal component of flux density is discontinuous across the surface by the amount of the surface charge density.

Consider two dielectric media 1 and 2 separated by a charge free boundary as shown in Fig.2.19.

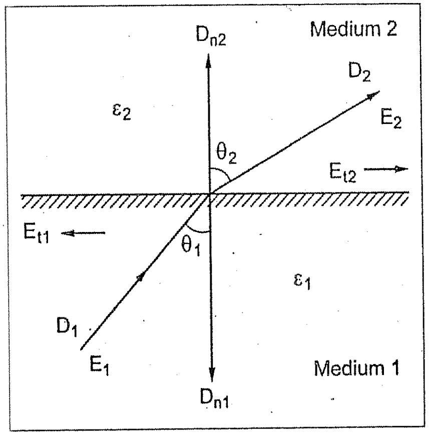


Fig. 2.19. Reflection and Refraction at the boundaries

Electric field intensity E_1 is incident in medium 1 at an angle of θ_1 and E_2 is the refracted electric field in medium 2 with an angle of θ_2. Similarly D_1 is incident electric displacement in medium 1 and D_2 is refracted electric displacement in medium 2.

The normal components of D are given below:

$$D_{n1} = D_1 \cos\theta_1$$
$$D_{n2} = D_2 \cos\theta_2$$

The tangential components of E are given below.

$$E_{t1} = E_1 \sin\theta_1$$
$$E_{t2} = E_2 \sin\theta_2$$

where θ_1 and θ_2 are incident angle and refracted angle respectively.

Apply the boundary conditions,

$$D_{n1} = D_{n2}$$
$$E_{t1} = E_{t2}$$
$$D_1 \cos\theta_1 = D_2 \cos\theta_2$$
$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

Dividing, $$\frac{E_1}{D_1} \tan\theta_1 = \frac{E_2}{D_2} \tan\theta_2$$

But $D_1 = \varepsilon_1 E_1$ and $D_2 = \varepsilon_2 E_2$

Then, $$\frac{E_1}{\varepsilon_1 E_1} \tan\theta_1 = \frac{E_2}{\varepsilon_2 E_2} \tan\theta_2$$

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

SOLVED EXAMPLES

Example 2.1

(i) *Given a field* $E = \left(\frac{-6y}{x^2}\right)\hat{I}_x + \left(\frac{6}{x}\right)\hat{I}_y + 5\,\hat{I}_z$ *V/m. Find the potential difference* V_{AB} *given* $A(-7, 2, 1)$ *and* $B(4, 1, 2)$.

(ii) *The electric field between two concentric cylindrical conductors at* $r = 0.01$ *m and* $r = 0.05$ *m is given by* $E = (10^6/r)\,\hat{I}_r$ *V/m. Find the electric potential.*

☺Solution:

(i) $$E = \left(\frac{-6y}{x^2}\right)\hat{I}_x + \left(\frac{6}{x}\right)\hat{I}_y + 5\,\hat{I}_z \quad \text{V/m}$$

Potential at point V $= -\int E \cdot dr$

$$V = -\left[\int \frac{-6y}{x^2}\,dx + \int \frac{6}{x}\,dy + \int 5\,dz\right]$$

$$= -\left[\frac{6y}{x} + \frac{6y}{x} + 5z\right]$$

Potential at a point A (– 7, 2, 1)

$$V_A = \left[\frac{2 \times 6 \times 2}{-7} + 5\right] = -1.57\ V$$

Potential at a point B (4, 1, 2)

$$V_B = -\left[\frac{2 \times 6 \times 1}{4} + 5 \times 2\right] = -13\ V$$

Potential difference between two points A and B is

$$V_{AB} = V_A - V_B = -1.57 - (-13) = \mathbf{11.43\ V}$$

(*ii*) $$E = \left(\frac{10^6}{r}\right)\hat{I}_r\ \ V/m$$

$$r = 0.01\ m$$

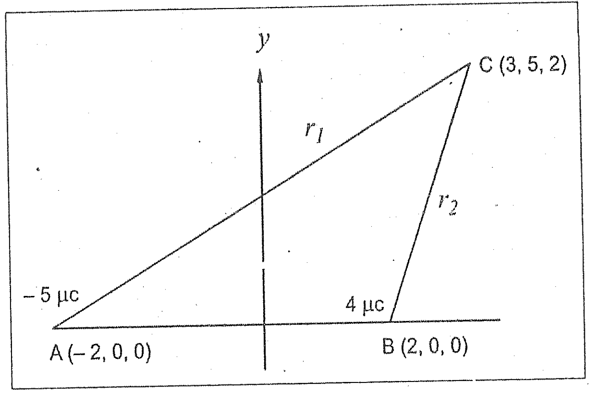
Electric potential $V = -\int E \cdot dr$

$$= \int_{0.01}^{0.05} \left(\frac{10^6}{r}\right) dr = -10^6\ ln\ r\,\Big|_{0.01}^{0.05}$$

$$= \mathbf{1.61 \times 10^6\ V}$$

☺☺

Example 2.2 *Find the potential at a point (3, 5, 2) due to two point charges, one located at (2, 0, 0) and the other at (–2, 0, 0). The charges are 4 μc and – 5 μc respectively. (All distances are in meters).*



☺**Solution:** $$r_1 = \sqrt{(3-(-2))^2 + (5-0)^2 + (2-0)^2} = 7.35\ m$$

$$r_2 = \sqrt{(3-2)^2 + (5-0)^2 + (2-0)^2} = 5.48\ m$$

Potential at C due to charge at A, $V_A = \dfrac{Q}{4\pi\,\varepsilon_0\,r_1}$

$$= \frac{-5 \times 10^{-6}}{4\pi \times \dfrac{1}{36\pi \times 10^9} \times 7.35} = -6122\ V$$

Potential at C due to charge at B, $V_B = \dfrac{4 \times 10^{-6}}{4\pi \times \dfrac{1}{36\pi \times 10^9} \times 5.48} = 6569\ V$

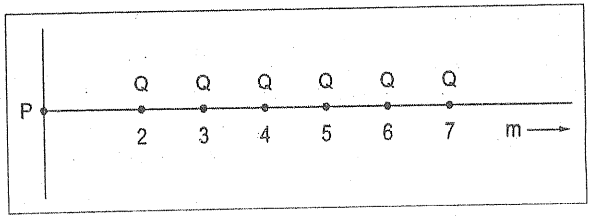
Potential at C due to charges at A and B

$$V = V_A + V_B$$

$$= 6569 - 6122 = 447\ V$$

☺☺

Example 2.3 *Six equal point charges, Q = 10 nc are located at 2, 3, 4, 5, 7 m. Find the potential at the origin.*



☺Solution: The potential at the origin is given by

$$V = \frac{1}{4\pi \varepsilon_0} \sum_{k=1}^{n} \frac{Q_n}{r_n}$$

$$V = \frac{10 \times 10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right]$$

$$= 90 \,[\, 1.593 \,] = \mathbf{143.35\ V}$$

☺☺

Example 2.4 *It is determined that the electric potential (scalar) gradient is, $3\,[(x^2 - yx)\,\hat{i} + (y^2 - 3\,x)\,\hat{j} + (z^2 - xy)\,\hat{k}\,]$ V/m, the potential at a point (1, 1, 1) in the rectangular co-ordinate system is found to be null.*

Determine the potential at any point (x, y, z) in the electric field whose parameters are stated above.

☺Solution: $\nabla V = 3\,[\,(x^2 - yx)\,\hat{i} + (y^2 - 3\,x)\,\hat{j} + (z^2 - xy)\,\hat{k}\,]$

Potential at (1, 1, 1) = 0

$$\nabla V = \left[\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \right] V$$

$$= 3\,[\,(x^2 - yx)\,\hat{i} + (y^2 - 3\,x)\,\hat{j} + (z^2 - x\,y)\,\hat{k}\,]$$

$$\therefore\ 3\,(x^2 - yz) = \frac{\partial V}{\partial x}$$

$$V = 3 \int (x^2 - yz)\, dx$$

$$V = 3 \left[\frac{x^3}{3} - xyz + C_1 \right]$$

$$3\,(y^2 - zx) = \frac{\partial V}{\partial y}$$

$$V = 3 \int (y^2 - zx)\, dy$$

$$V = 3 \left[\frac{y^3}{3} - yxz + C_2 \right]$$

$$3\,(z^2 - yx) = \frac{\partial V}{\partial z}$$

$$V = 3 \int (z^3 - yx)\, dz$$

$$V = 3 \left[\frac{z^3}{3} - xyz + C_3 \right]$$

Adding the values of all V

$$\therefore\ V = x^3 + y^3 + z^3 - 9\,xyz + C_1 + C_2 + C_3$$

$$V = (1,\,1,\,1) = 0, \text{ then}$$

$$V = 1 + 1 + 1 - 9 + C_1 + C_2 + C_3 = 0$$

$$0 = -6 + C_1 + C_2 + C_3$$

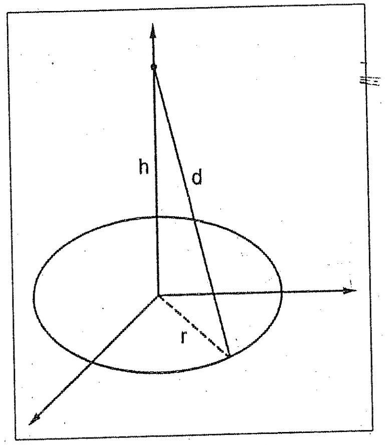
$$\therefore\ C_1 + C_2 + C_3 = 6$$

Substituting the given condition, potential at any point

$$V = \mathbf{x^3 + y^3 + z^3 - 9\,xyz + 6}$$

☺☺

Example 2.5 *A total charge (40/3) nano coulomb is uniformly distributed around a circular ring of radius 1 m. Find the potential at a point on the axis 5 m from the plane of ring.*



☺Solution:

$$q = \frac{40}{3}\ \text{nC}$$

Radius of ring, $r = 1$ m

Distance of point, $h = 5$ m

$$d = \sqrt{1^2+5^2} = \sqrt{26}$$

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{\frac{40}{3}\times 10^{-9}}{4\pi \times 8.854\times 10^{-12}\times\sqrt{26}}$$

$$= 23.5 \text{ V}$$

∴ **The potential at a point on the axis 5 m from the plane of ring is 23.5 volts.** ☺☺

Example 2.6 *Given a field* $E = \frac{-6y}{x^2}\bar{a}_x + \frac{6}{x}\bar{a}_y + 5\bar{a}_z$ *V/m, find the potential difference* V_{AB} *between A (– 7, 2, 1) and B (4, 1, 2).*

☺ ***Solution:*** $E = \frac{-6y}{x^2}\bar{a}_x + \frac{6}{x}\bar{a}_y + 5\bar{a}_z$

Potential difference $V_{AB} = -\int_B^A E\cdot dl$

$$= -\int_4^{-7} -\frac{6y}{x^2}\,dx - \int_1^2 \frac{6}{x}\,dy - \int_2^1 5\,dx$$

$$= 6y\left[\frac{-1}{x}\right]_4^{-7} - \frac{6}{x}\left[y\right]_1^2 - 5\left[x\right]_2^1$$

$$= \frac{66}{28}y - \frac{6}{x} + 5$$

$$x = 4, \quad y = 1$$

$V_{AB} = 2.357 - 1.5 + 5 =$ **5.857 volts** ☺☺

Example 2.7 *If* $V = \frac{20}{x^2+y^2}$, *find E and D.*

☺***Solution:*** $E = -\nabla V$

$$= -20\left[\frac{\partial}{\partial x}(x^2+y^2)^{-1} + \frac{\partial}{\partial y}(x^2+y^2)^{-1} + \frac{\partial}{\partial z}(x^2+y^2)^{-1}\right]$$

$$= -20\left[\frac{-2x}{(x^2+y^2)^2} - \frac{2y}{(x^2+y^2)^2}\right]$$

$$E = 40\left[\frac{x}{(x^2+y^2)^2} + \frac{y}{(x^2+y^2)^2}\right]$$

$$D = \varepsilon E$$

$$= 40\,\varepsilon\left[\frac{x}{(x^2+y^2)^2} + \frac{y}{(x^2+y^2)^2}\right]$$ ☺☺

Example 2.8 *In a rectangular co-ordinate referenced system, the electric potential is,* $V = \frac{20}{x^2+y^2+z^2}$

Determine the electric vector that is the gradient of potential.

☺***Solution:*** $V = \frac{20}{x^2+y^2+z^2}$

$$E = -\nabla\cdot\bar{V} = -\left(\frac{\partial}{\partial x}\frac{20}{x^2+y^2+z^2}\bar{a}_x + \frac{\partial}{\partial y}\frac{20}{x^2+y^2+z^2}\bar{a}_y + \frac{\partial}{\partial z}\frac{20}{x^2+y^2+z^2}\bar{a}_z\right)$$

$$= \frac{40}{(x^2+y^2+z^2)^2}\left[x\,\bar{a}_x + y\,\bar{a}_y + z\,\bar{a}_z\right] \text{ V/m}$$

☺☺

Example 2.9 *If* $V = 2x^2y + 20z - \frac{4}{x^2+y^2}$ *volts, find* $\bar{E}$ *and* $\bar{D}$ *at* $\rho(6, -2.5, 3)$.

☺***Solution:*** $E = -\nabla V$

$$= -\left[\bar{a}_x\frac{\partial}{\partial x} + \bar{a}_y\frac{\partial}{\partial y} + \bar{a}_z\frac{\partial}{\partial z}\right]\left(2x^2y + 20z - \frac{4}{x^2+y^2}\right)$$

$$= -\left[\bar{a}_x\left(4xy + \frac{8x}{(x^2+y^2)^2}\right) + \bar{a}_y\left(2x^2 + \frac{8y}{(x^2+y^2)^2}\right) + \bar{a}_z 20\right]$$

$$E(6, -2.5, 3) = -[(-60 + 0.0268)\bar{a}_x + (72 - 0.012)\bar{a}_y + 20\bar{a}_z]$$

$$= \mathbf{59.97\,\bar{a}_x - 71.99\,\bar{a}_y - 20\,\bar{a}_z \text{ V/m}}$$

$$D = \varepsilon_0 E$$

$$D(6, -2.5, 3) = 8.854 \times 10^{-12} \times [59.97\,\bar{a}_x - 71.99\,\bar{a}_y - 20\,\bar{a}_z]$$

$$= \mathbf{0.53\,\bar{a}_x - 0.637\,\bar{a}_y - 0.177\,\bar{a}_z \ \ nc/m^2}$$ ☺☺

Example 2.10 *Given* $V = \frac{120 \sin \theta}{r^2}$ *in free space. Find the electric potential and field intensity at the point located at* $r = 3$ m, $\theta = 60°$ *and* $\phi = 25°$.

☺***Solution:*** Electric potential $V = \frac{120 \sin \theta}{r^2} = \frac{120 \sin 60°}{3^2} = \mathbf{11.547\ V}$

Electric field intensity $E = -\nabla V$

For spherical co-ordinates,

$$\nabla V = \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r \sin \theta}\frac{\partial V}{\partial \phi}\bar{a}_\phi$$

$$\frac{\partial V}{\partial r} = \frac{-240 \sin \theta}{r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{120 \cos \theta}{r^2}$$

$$\frac{\partial V}{\partial \phi} = 0$$

$$\nabla V = \frac{-240 \sin \theta}{r^3}\bar{a}_r + \frac{120 \cos \theta}{r^3}\bar{a}_\theta$$

$$E = -\nabla V = \frac{240 \sin 60°}{3^3}\bar{a}_r - \frac{120 \cos 60°}{3^3}\bar{a}_\theta$$

$$= 7.7\,\bar{a}_r - 2.22\,\bar{a}_\theta$$

$$|E| = \sqrt{(7.7)^2 + (2.22)^2} = \mathbf{8.014\ V/m}$$ ☺☺

Example 2.11 *Given* $V = x^2 y + 10z + 2 \log (x^2 + y^2)$ *find V, E, D and* ρ_v *at (1, 2, 3).*

☺***Solution:***

(i) $V = x^2 y + 10z + 2 \log (x^2 + y^2)$

$$V(1, 2, 3) = 1 \times 2 + 10 \times 3 + 2 \log (1 + 4) = 2 + 30 + 2 \log (5)$$

$$= 32 + 2 \log 5 = \mathbf{33.4\ V}$$

(ii) $E = -\nabla V$

$$= -\left[\bar{a}_x \frac{\partial}{\partial x}(x^2 y + 2 \log (x^2 + y^2)) + \bar{a}_y \frac{\partial}{\partial y}(x^2 y + 2 \log (x^2 + y^2)) + \bar{a}_z \frac{\partial}{\partial z}(10z)\right]$$

$$= -\left[\bar{a}_x \left(2xy + \frac{4x}{x^2 + y^2}\right) + \bar{a}_y \left(x^2 + \frac{4y}{x^2 + y^2}\right) + \bar{a}_z (10)\right]$$

$$E(1, 2, 3) = -\left[\left(4 + \frac{4}{1+4}\right)\bar{a}_x + \left(1 + \frac{8}{1+4}\right)\bar{a}_y + 10\,\bar{a}_z\right]$$

$$= -\left[\frac{24}{5}\bar{a}_x + \frac{13}{5}\bar{a}_y + 10\,\bar{a}_z\right]$$

$$= \mathbf{-(4.8\,\bar{a}_x + 2.6\,\bar{a}_y + 10\,\bar{a}_z)\ \ V/m}$$

(iii) $D = \varepsilon \cdot E$

$$D = \varepsilon_0\left[\left(2xy + \frac{4x}{x^2 + y^2}\right)\bar{a}_x + \left(x^2 + \frac{4y}{x^2 + y^2}\right)\bar{a}_y + 10\,\bar{a}_z\right]$$

$$D(1, 2, 3) = \mathbf{-8.854 \times 10^{-12}\ (4.8\,\bar{a}_x + 2.6\,\bar{a}_y + 10\,\bar{a}_z)\ \ c/m^2}$$

(iv) $\rho_v = \nabla \cdot D$

$$= \left[\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}\right] D$$

But $D = \varepsilon_0 E$

$$\nabla \cdot D = -\varepsilon_0 \left[\bar{a}_x \frac{\partial}{\partial x} + \bar{a}_y \frac{\partial}{\partial y} + \bar{a}_z \frac{\partial}{\partial z}\right] \times \left[\bar{a}_x \left(2xy + \frac{4x}{x^2 + y^2}\right) + \bar{a}_y \left(x^2 + \frac{4y}{x^2 + y^2}\right) + \bar{a}_z (10)\right]$$

$$= -\varepsilon_0 \left[\frac{\partial}{\partial x}\left(2xy + \frac{4x}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(x^2 + \frac{4y}{x^2 + y^2}\right) + \frac{\partial}{\partial z}(10)\right]$$

$$= -\varepsilon_0 \left[2y + \frac{(x^2+y^2)\,4 - 4x\,(2x)}{x^2+y^2} + \frac{(x^2+y^2)\,4 - 4y\,(2y)}{x^2+y^2} + 0 \right]$$

$$= -\varepsilon_0 \left[2y + \frac{4y^2 - 4x^2}{x^2+y^2} + \frac{4x^2 - 4y^2}{x^2+y^2} \right]$$

$$= -\varepsilon_0 \left[2y + \frac{4y^2 - 4x^2 + 4x^2 - 4y^2}{x^2+y^2} \right]$$

$$\rho_v = -\varepsilon_0\, 2y$$

$$\rho_v(1, 2, 3) = -8.854 \times 10^{-12} \times 2 \times 2$$

$$= \mathbf{35.416 \times 10^{-12} \quad c/m^3}$$

☺☺

Example 2.12 *Given:* $V = \dfrac{120 \sin\theta}{r^2}$

Find the E and V at a point located at $r = 3$ m, $\theta = 60°$, $\phi = 25°$.

☺Solution: $E = -\nabla \cdot \nabla = -\left(\dfrac{\partial V}{\partial r}\,\bar{a}_r + \dfrac{\partial V}{r\,\partial\theta}\bar{a}_\theta + \dfrac{1}{r\sin\theta}\,\dfrac{\partial V}{\partial\phi}\bar{a}_\phi \right)$

$$= -\left(\frac{-2}{r^3}\,(120\sin\theta)\,\bar{a}_r + \frac{1}{r}\,\frac{120\cos\theta}{r^2}\,\bar{a}_\theta \right)$$

$$= \frac{1}{r^3}\,[\,240\sin\theta\,\bar{a}_r - 120\cos\theta\cdot\bar{a}_\theta\,]$$

$$= \frac{120}{r^3}\,[\,2\sin\theta\;\bar{a}_r - \cos\theta\;\bar{a}_\theta\,]$$

$$E\,(3, 60°, 25°) = \frac{120}{27}\,[\,2\sin 60°\;\bar{a}_r - \cos 60°\;\bar{a}_\theta\,]$$

$$= \frac{120}{27}\,[\,\sqrt{3}\;\bar{a}_r - \frac{1}{2}\bar{a}_\theta\,] \quad V/m$$

$$= [\,\mathbf{7.698\,\bar{a}_r - 2.222\,\bar{a}_\theta}\,] \quad \mathbf{V/m}$$

$$V = \frac{120\sin 60°}{9} = \mathbf{11.55 \;\; volts}$$

☺☺

Example 2.13 *If $V = \dfrac{60\sin\theta}{r^2}$ volts, find V and E at P(3, 60°, 25°) where V is the electric potential and E is the electric field intensity.*

☺Solution: $V = \dfrac{60\sin\theta}{r^2}$

V at P(3, 60°, 25°)

$$V = \frac{60\sin 60°}{3^2} = 5.77\text{ V}$$

For spherical co-ordinates

$$E = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial r}\,\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\,\bar{a}_\theta + \frac{1}{r\sin\theta}\,\frac{\partial V}{\partial\phi}\,\bar{a}_\phi \right]$$

$$\frac{\partial V}{\partial r} = 60\sin\theta\,\frac{\partial}{\partial r}\,\frac{1}{r^2} = -\frac{120\sin\theta}{r^3}$$

$$\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{60}{r^3}\,\frac{\partial}{\partial\theta}\,(\sin\theta) = \frac{60\cos\theta}{r^3}$$

$$\frac{\partial V}{\partial\phi} = 0$$

$$E = \frac{120\sin\theta}{r^3}\cdot\bar{a}_r - \frac{60\cos\theta}{r^3}\cdot\bar{a}_\theta$$

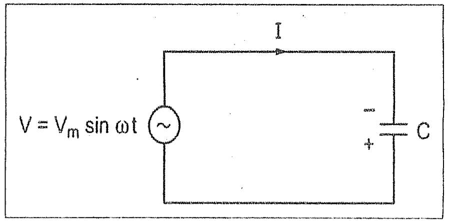
E at P(3, 60°, 25°)

$$E = \frac{120\sin 60}{3^3}\cdot\bar{a}_r - \frac{60\cos 60}{3^3}\cdot\bar{a}_\theta$$

$$= \mathbf{3.85\,\bar{a}_r - 1.11\,\bar{a}_\theta \quad V/m.}$$

☺☺

Example 2.14 *Show that the displacement current through the parallel plate capacitor shown in Fig., is equal to the conduction current I flowing in the external circuit.*



☺ **Solution:** The displacement current

$$I_D = \int_s J_D \cdot ds = \int_s \frac{\partial D}{\partial t}\, ds = \frac{\partial D}{\partial t}\int_s ds = \frac{\partial D}{\partial t}\cdot A$$

$$\text{But} \quad D = \varepsilon E$$

$$E = \frac{D}{\varepsilon}$$

E is the electric field exists in the capacitor when voltage V is applied across the capacitor,

$$E = \frac{V}{d}$$

where, d is the distance between two plates.

$$\text{Then,} \quad E = \frac{D}{\varepsilon} = \frac{V}{d}$$

$$D = \frac{\varepsilon V}{d}$$

Differentiating with respect to 't'

$$\frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{\partial V}{\partial t}$$

$$\text{Let} \quad V = V_m \sin \omega t$$

$$\frac{\partial V}{\partial t} = V_m\, \omega \cos \omega t$$

Substituting $\frac{\partial V}{\partial t}$ in $\frac{\partial D}{\partial t}$ equation

$$\frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\,\omega V_m \cos \omega t$$

$$I_D = A \cdot \frac{\varepsilon}{d}\,\omega V_m \cos \omega t$$

For conduction current, $\quad I_C = C\,\frac{dV}{dt}$

$$\text{But} \quad C = \varepsilon \frac{A}{d} \quad \text{and}$$

$$\frac{dV}{dt} = \omega V_m \cos \omega t$$

Conduction current, $\quad I_C = \frac{\varepsilon A}{d}\,\omega V_m \cos \omega t$

Hence, $\quad \boldsymbol{I_D = I_C}$ ☺☺

Example 2.15 *If a potential $v = x^2yz + A\,y^3z$ find the value of A so that Laplace's equation is satisfied and electric field at (2, – 2, 1).*

☺**Solution:** $\quad V = x^2\,y\,z + A\,y^3\,z$

$$\text{Laplace's equation} \quad \nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial V}{\partial x} = 2\,x\,y\,z \quad \text{and} \quad \frac{\partial^2 V}{\partial x^2} = 2\,y\,z$$

$$\frac{\partial V}{\partial y} = x^2\,z + 3\,A\,y^2\,z \quad \text{and} \quad \frac{\partial^2 V}{\partial y^2} = 6\,A\,y\,z$$

$$\frac{\partial V}{\partial z} = x^2\,y + A\,y^3 \quad \text{and} \quad \frac{\partial^2 V}{\partial z^2} = 0$$

∴ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2\,y\,z + 6\,A\,y\,z = 0$$

At (2, – 2, 1),

$$2\,(-2)\,1 + 6\,A\,(-2)\,1 = 0$$

$$-4 - 12\,A = 0$$

$$A = -1/3$$

$$\therefore\; V = x^2\,y\,z - \frac{y^3\,z}{3}$$

$$\text{Electric field } E = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial x}\,\bar{a}_x + \frac{\partial V}{\partial y}\,\bar{a}_y + \frac{\partial V}{\partial z}\,\bar{a}_z\right]$$

$$= -\left[\,(2\,x\,y\,z)\,\bar{a}_x + (x^2\,z - y^2\,z)\,\bar{a}_y + (x^2\,y - \frac{y^3}{3})\;\bar{a}_z\,\right]$$

$$E(2,-2,1) = -[2\,(2)\,(-2)\,\bar{a}_x + (2^2 - (-2)^2)\,\bar{a}_y + (2^2\,(-2) - \frac{(-2)^3}{3})\;\bar{a}_z\,]$$

$$= \boldsymbol{8\,\bar{a}_x + \frac{32}{3}\,\bar{a}_z}$$ **Ans.** ☜ ☺☺

Example 2.16 *A parallel plate air core capacitor is floating with 10 V across the plates. If the separation between the plates is now increased to double the original value, calculate the new value of the voltage across the capacitor.*

☺***Solution:*** $Q = CV, \quad V = \frac{Q}{C}, \quad C = \frac{\varepsilon_0 A}{d}$

If d becomes double, then new capacitance

$$C' = \frac{\varepsilon_0 A}{2d}, \quad C = \frac{C}{2}$$

$$V' = \frac{Q}{C} = \frac{2Q}{C}$$

$$V' = 2V$$

$$V' = 2 \times 10 = \mathbf{20\ V} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.17 *A flat slab of dielectric with relative permittivity 5 is placed normal to a uniform field of flux density 2 c/m². The slab occupies a volume of 0.1 m³ and is uniformly polarized. Find (a) Polarization in the slab and (b) total dipole moment of the slab.*

☺ ***Solution:***

(a)

$$D = \varepsilon_0 E + P$$

$$P = D - \varepsilon_0 E = D - \frac{D}{\varepsilon_r} \qquad [D = \varepsilon_0 \varepsilon_r E]$$

$$= D\left[1 - \frac{1}{\varepsilon_r}\right] = \left[1 - \frac{1}{5}\right]$$

$$= 2\left[\frac{5-1}{5}\right] = \mathbf{1.6\ c/m^2} \quad \textbf{Ans.}$$ ☜

(b) Total dipolement = Polarization × Volume

$$= 1.6 \times 0.1$$

$$= \textbf{0.16 coulomb metre} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.18 *A capacitor has two large horizontal parallel plates spaced 't' apart. A dielectric slab of relative permittivity ε_r and thickness t' is now placed*

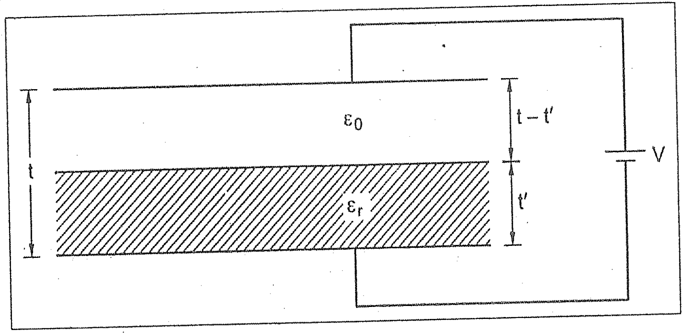
(between the plates) on the lower plate. A p.d. of V is applied across the capacitor plates. Show that the electric field E_d in the dielectric is given by

$$E_d = \frac{V}{\varepsilon_r t - t'(\varepsilon_r - 1)}$$

and that the electric field in the air space between the top of the dielectric slab and the upper capacitor plate is

$$E_g = \frac{\varepsilon_r V}{\varepsilon_r t - t'(\varepsilon_r - 1)}$$

☺ ***Solution:***



The potential V is applied across the capacitor. Let V_d and V_g are the potentials across the dielectric and air medium respectively.

Then $V = V_d + V_g$

But $V_d = E_d t'$

$$V_g = E_g (t - t')$$

$$V = E_d t' + E_g (t - t')$$

The electric flux density D is the same for both the media. Then

$$E_d = \frac{D}{\varepsilon_r \varepsilon_0}$$

$$E_g = \frac{D}{\varepsilon_0}$$

Substituting these values in the above equation,

$$V = \frac{D}{\varepsilon_0}\left[\frac{t'}{\varepsilon_r} + t - t'\right]$$

$$D = \frac{V\varepsilon_0}{\frac{t'}{\varepsilon_r} + t - t'}$$

Substituting the value of D in E_d equation

$$E_d = \frac{\frac{V\varepsilon_0}{\left[\frac{t'}{\varepsilon_r} + t - t'\right]}}{\varepsilon_0\varepsilon_r} = \frac{V}{t' + (t - t')\,\varepsilon_r} = \frac{V}{\varepsilon_r t + t'(1 - \varepsilon_r)}$$

$$E_d = \frac{V}{\varepsilon_r t + t'(\varepsilon_r - 1)}$$ **Ans.** ☜

and similarly for E_g.

$$E_g = \frac{\frac{V\varepsilon_0}{\left[\frac{t'}{\varepsilon_r} + t - t'\right]}}{\varepsilon_0} = \frac{V}{\frac{t'}{\varepsilon_r} + t - t'} = \frac{V\varepsilon_r}{t' + \varepsilon_r(t - t')}$$

$$E_g = \frac{V\varepsilon_r}{\varepsilon_r t - t'(\varepsilon_r - 1)}$$ **Ans.** ☜ ☺☺

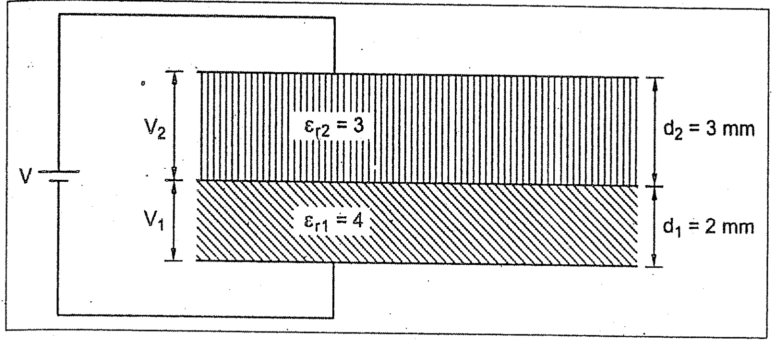
Example 2.19 *Calculate the capacitance of a parallel plate capacitor with the following details:*

Plate area $A = 100\,cm^2$

Dielectric 1, $\varepsilon_{r_1} = 4, \quad d_1 = 2\,mm$

Dielectric 2, $\varepsilon_{r_2} = 3, \quad d_2 = 3\,mm$

If 200 V is applied across the plates, what will be the voltage gradient across each dielectric.



☺ **Solution:**

$$A = \text{Area} = 0.01\ m^2$$

$$\varepsilon_0 = 8.854 \times 10^{-12}\ F/m$$

If potential V is applied across the capacitor

$$V = V_1 + V_2$$

$$V_1 = E_1 d_1$$

$$V_2 = E_2 d_2$$

Flux density $D = \frac{Q}{A}$ is same for the media.

The electric field intensity in medium 1 and medium 2.

$$E_1 = \frac{D}{\varepsilon_0\varepsilon_{r_1}} = \frac{Q}{A\varepsilon_0\varepsilon_{r_1}}$$

$$E_2 = \frac{D}{\varepsilon_0\varepsilon_{r_2}} = \frac{Q}{A\varepsilon_0\varepsilon_{r_2}}$$

Applied potential $V = V_1 + V_2 = \frac{Q d_1}{A\varepsilon_0\varepsilon_{r_1}} + \frac{Q d_2}{A\varepsilon_0\varepsilon_{r_2}}$

$$V = \frac{Q}{A\varepsilon_0}\left[\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}}\right]$$

Capacitance $C = \frac{Q}{V} = \frac{A\varepsilon_0}{\left[\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}}\right]}$

$$= \frac{0.01 \times 8.854 \times 10^{-12}}{\frac{2 \times 10^{-3}}{4} + \frac{3 \times 10^{-3}}{3}}$$

$$= 0.059 \text{ nF} = \mathbf{59\ pF} \quad \textbf{Ans.}$$

$$\frac{V_1}{V_2} = \frac{d_1/\varepsilon_{r_1}}{d_2/\varepsilon_{r_2}} = \frac{2 \times 10^{-3}/4}{3 \times 10^{-3}/3} = \frac{1}{2}$$

$$V_1 = \frac{V_2}{2}$$

$$V_1 + V_2 = 200 \text{ V (given)}$$

$$\frac{V_2}{2} + V_2 = 200$$

$$\frac{3\,V_2}{2} = 200$$

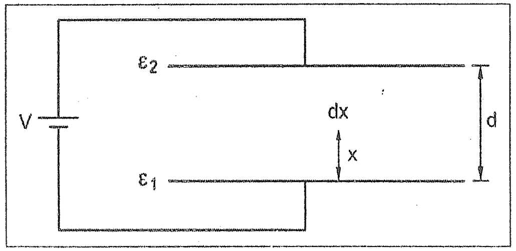
$$V_2 = \frac{400}{3} = \mathbf{133.33\ V} \quad \textbf{Ans.}$$

$$V_1 = \mathbf{66.66\ V} \quad \textbf{Ans.}$$

$$E_1 = \frac{V_1}{d_1} = \frac{66.66}{2 \times 10^{-3}} = \mathbf{33.33\ kV/m} \quad \textbf{Ans.}$$

$$E_2 = \frac{V_2}{d_2} = \frac{133.33}{3 \times 10^{-3}} = \mathbf{44.44\ kV/m} \quad \textbf{Ans.}$$

Example 2.20 *The permittivity of the dielectric of a parallel plate capacitor increases uniformly from ε_1 at one plate to ε_2 at the other. If A is the surface area of the plate and d is the thickness of dielectric, derive an expression for capacitance.*



☺ ***Solution:*** The permittivity of any point in the dielectric medium at a distance 'x' from the lower plate is

$$\varepsilon = \varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x$$

Electric field at the point if D is the flux density is

$$E = \frac{D}{\varepsilon} = \frac{D}{\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x}$$

Electric potential at the point $\quad V = -E\,dx = -\dfrac{D}{\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x}\,dx$

The total electric potential across the capacitor

$$V = -\int E\,dx = -\int_0^d \frac{D}{\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} dx}\,dx$$

$$= \frac{D}{\varepsilon_2 - \varepsilon_1}\left[ln\left(\varepsilon_1 + \frac{\varepsilon_2 - \varepsilon_1}{d} x\right)_0^d = \frac{D}{\varepsilon_2 - \varepsilon_1}[ln\,\varepsilon_2 - ln\,\varepsilon_1]\right.$$

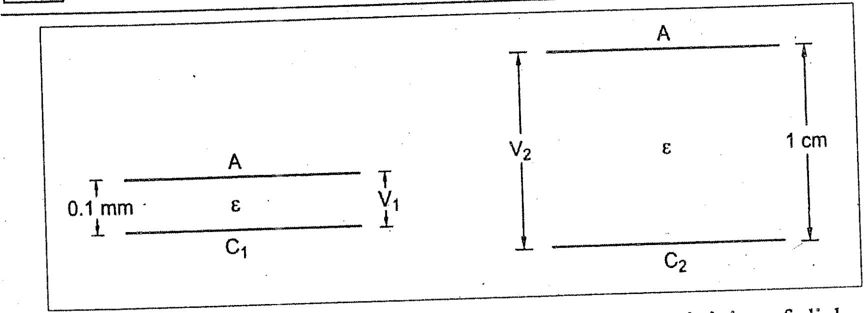
$$= \frac{D}{\varepsilon_2 - \varepsilon_1} ln\left[\frac{\varepsilon_2}{\varepsilon_1}\right]$$

Capacitance / unit area $\quad C = \dfrac{Q}{A\,V} = \dfrac{D}{V} = \dfrac{\varepsilon_2 - \varepsilon_1}{ln\left[\frac{\varepsilon_2}{\varepsilon_1}\right]}$

$$C = \frac{\varepsilon_2 - \varepsilon_1}{\mathbf{ln}\left[\frac{\varepsilon_2}{\varepsilon_1}\right]} \ \mathbf{F/m^2} \quad \textbf{Ans.}$$

☺☺

Example 2.21 *Two parallel charged metal plates are separated by a distance of 0.1 mm and charged to a p.d of 100 V. If the charge is not allowed to leak off and the plates are now separately by 1 cm, what is the new p.d between them?*



☺ ***Solution:*** If A is the area of the plate and ε is permittivity of dielectric medium, the capacitance of old capacitor

$$C_1 = \frac{\varepsilon A}{d} = \frac{\varepsilon A}{0.1 \times 10^{-3}}$$

The capacitance of new capacitor

$$C_2 = \frac{\varepsilon A}{1 \times 10^{-2}} \qquad (d = 1 \text{ cm})$$

Since charge is the same for both capacitors

$$Q = C_1 V_1 = C_2 V_2$$

$$C_2 V_2 = C_1 V_1$$

$$V_2 = \frac{C_1}{C_2} V_1$$

$$= \frac{1 \times 10^{-2}}{0.1 \times 10^{-3}} V_1 = 10^2 \times 100 \qquad [\because V_1 = 100 \text{ V}]$$

$$\mathbf{V_2 = 10 \text{ kV}} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.22 *A parallel plate capacitor with d = 1 m and plate area 0.8 m^2 and a dielectric relative permittivity of 2.8. A dc volt of 500 V is applied between the plates, find the capacitance and energy stored.*

☺ ***Solution:*** Capacitance $C = \frac{\varepsilon A}{d}$

$$d = 1 \text{ m}$$

$$A = 0.8 \text{ m}^2$$

$$\varepsilon = \varepsilon_0 \varepsilon_r = 8.854 \times 10^{-12} \times 2.8$$

$$C = \frac{8.854 \times 10^{-12} \times 2.8 \times 0.8}{1}$$

$$\mathbf{C = 19.833 \text{ pF}} \quad \textbf{Ans.}$$ ☜

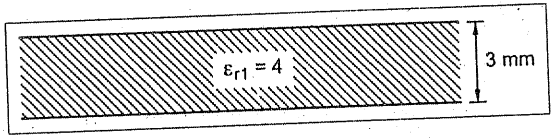
$$\text{Energy stored} = \frac{1}{2} C V^2$$

$$= \frac{1}{2} \times 19.833 \times (500)^2 \times 10^{-12}$$

$$\mathbf{E = 2.48 \times 10^{-6} \text{ Joules}} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.23 *A condenser is composed of two plates separated by a sheet of insulating material 3 mm thick and of permittivity 4. The distance between the plates is increased to allow the insertion of a second sheet 5 mm thick of permittivity K. If the capacitance of the condenser so formed is 1/3 of the original capacitance, find K.*

☺ ***Solution:***

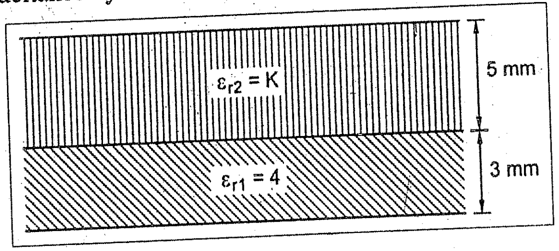


The capacitance of the conductor

$$C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0 \varepsilon_r A}{d}$$

$$C = \frac{4 \varepsilon_0 A}{3 \times 10^{-3}}$$

The new capacitance by insertion of another dielectric



$$C' = \frac{A\,\varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}}} = \frac{A\,\varepsilon_0}{\frac{3\times10^{-3}}{4} + \frac{5\times10^{-3}}{K}}$$

But $C' = \frac{1}{3}C$

$$= \frac{A\,\varepsilon_0}{\frac{3\times10^{-3}}{4} + \frac{5\times10^{-3}}{K}} = \frac{1}{3} \times \frac{4\,\varepsilon_0 A}{3\times10^{-3}} = \frac{1}{\frac{3}{4} + \frac{5}{K}} = \frac{4}{9}$$

$$= \frac{3}{4} + \frac{5}{K} = \frac{9}{4}$$

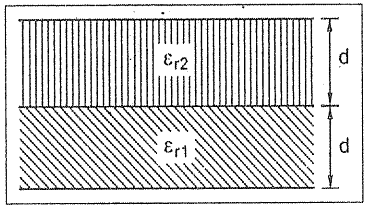
$$\frac{5}{K} = \frac{9}{4} - \frac{3}{4} = \frac{6}{4}$$

$$\frac{K}{5} = \frac{4}{6}$$

$$K = \frac{\mathbf{20}}{\mathbf{6}}$$ **Ans.** ☜

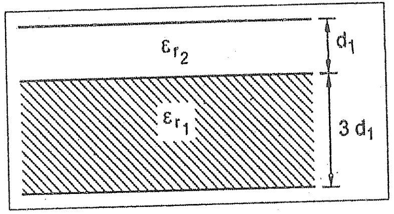
☺☺

Example 2.24 *A parallel plate capacitor has two dielectrics with relative permittivities ε_{r_1} and ε_{r_2}. If C_1 is the capacitance when two dielectrics take same thickness and C_2 is the capacitance when one dielectric takes thickness 3 times of other one, find $\frac{C_1}{C_2}$.*



☺ ***Solution:*** The capacitance C_1 having equal thickness of dielectrics.

$$C_1 = \frac{\varepsilon_0 A}{\frac{d}{\varepsilon_{r_1}} + \frac{d}{\varepsilon_{r_2}}}$$



The capacitance C_2 having unequal thickness of dielectrics.

$$C_2 = \frac{\varepsilon_0 A}{\frac{3\,d_1}{\varepsilon_{r_1}} + \frac{d_1}{\varepsilon_{r_2}}}$$

$$\frac{C_1}{C_2} = \frac{\dfrac{\varepsilon_0 A}{\frac{d}{\varepsilon_{r_1}} + \frac{d}{\varepsilon_{r_2}}}}{\dfrac{\varepsilon_0 A}{\frac{3\,d_1}{\varepsilon_{r_1}} + \frac{d_1}{\varepsilon_{r_2}}}} = \frac{\frac{3\,d_1}{\varepsilon_{r_1}} + \frac{d_1}{\varepsilon_{r_2}}}{\frac{d}{\varepsilon_{r_1}} + \frac{d}{\varepsilon_{r_2}}}$$

$$= \frac{d_1\left[\frac{3}{\varepsilon_{r_1}} + \frac{1}{\varepsilon_{r_2}}\right]}{d\left[\frac{1}{\varepsilon_{r_1}} + \frac{1}{\varepsilon_{r_2}}\right]} = \frac{d_1}{d}\left[\frac{3\,\varepsilon_{r_2} + \varepsilon_{r_1}}{\varepsilon_{r_2} + \varepsilon_{r_1}}\right]$$

$$= \frac{d_1}{d}\left[\frac{\varepsilon_{r_1} + 3\,\varepsilon_{r_2}}{\varepsilon_{r_1} + \varepsilon_{r_2}}\right] = \frac{d_1}{d}\left[\frac{1 + 3\frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}}{1 + \frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}}\right]$$

$$\frac{C_1}{C_2} = \frac{d_1}{d}\left[\frac{1 + 3\frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}}{1 + \frac{\varepsilon_{r_2}}{\varepsilon_{r_1}}}\right]$$ **Ans.** ☜

☺☺

Example 2.25 *The capacitance of a condenser formed by two parallel metal sheets, each 100 cm² in area separated by a dielectric 2 mm thick is 2×10^{-4} μF. A potential of 20 kV is applied. Find (a) the electric flux, (b) potential gradient in kV/cm, (c) the relative permittivity of the material and (d) the electric flux density.*

☺ **Solution:**

$$C = 2 \times 10^{-4} \times 10^{-6}\ F = 2 \times 10^{-10}\ F$$

$$d = 2 \times 10^{-3}\ m$$

$$A = 100\ cm^2$$

$$V = 20 \times 10^3\ V$$

(*a*) Flux

$$\chi = Q = CV$$

$$= 2 \times 10^{-10} \times 20 \times 10^3 = 4 \times 10^{-6}\ coulomb$$

$$\chi = \mathbf{4\ \mu C}$$ **Ans.** ☜

(*b*) Potential gradient (Electric field)

$$E = \frac{V}{d}$$

$$= \frac{20 \times 10^3}{2 \times 10^{-3}} = 10 \times 10^6\ V/m$$

$$E = \mathbf{10\ MV/m}$$ **Ans.** ☜

(*c*) $C = \varepsilon_0 \varepsilon_r A$

Relative permittivity

$$\varepsilon_r = \frac{C\ d}{\varepsilon_0 A}$$

$$= \frac{2 \times 10^{-10} \times 2 \times 10^{-3}}{8.854 \times 10^{-12} \times 1 \times 10^{-2}}$$

$$= \frac{4 \times 10^{-13} \times 10^{14}}{8.854} = 4.518$$

$$\varepsilon_r = \mathbf{4.518}$$ **Ans.** ☜

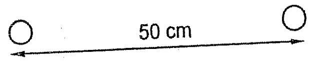
(*d*) Electric flux density,

$$D = \frac{\chi}{A}$$

$$= \frac{4 \times 10^{-3}}{1 \times 10^{-2}}$$

$$= \mathbf{0.4\ C/m^2}$$ **Ans.** ☜

Example 2.26 *Calculate the capacitance per km between a pair of parallel wires each of diameter 1 cm at a spacing of 50 cm.*



☺ **Solution:**

$$D = 0.5\ m$$

$$r = \frac{d}{2} = 0.5\ cm = 0.005\ m$$

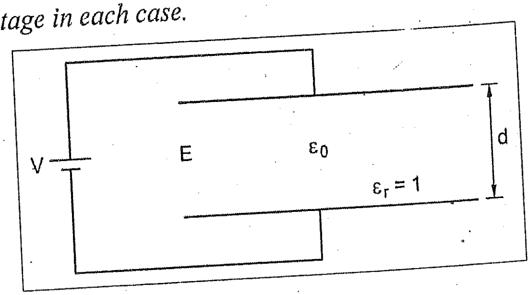
The capacitance between two parallel conductor is

$$C = \frac{\pi\ \varepsilon_0}{ln\ D/r} = \frac{\pi \times 8.854 \times 10^{-12}}{ln\left[\frac{0.5}{0.005}\right]} = 6 \times 10^{-12}\ F$$

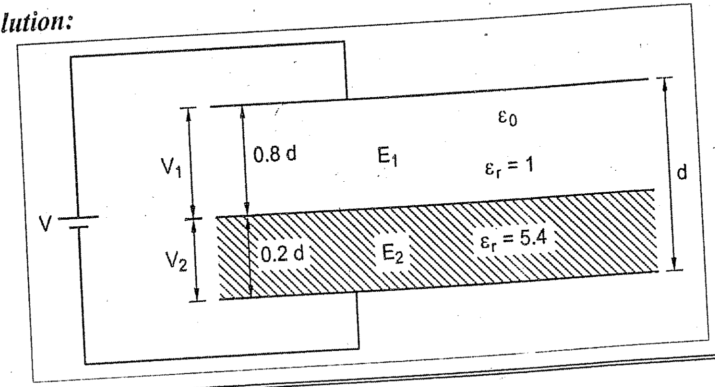
$$= \mathbf{6\ pF}$$ **Ans.** ☜

☺☺

Example 2.27 *A parallel plate capacitance has free space between the plates. Compare the voltage gradient in this case to that in the free space, when a sheet of mica, $\varepsilon_r = 5.4$, fills 20% of the distance between the plates. Assume the same applied voltage in each case.*



☺ **Solution:**



A parallel plate capacitor has free space ($\varepsilon_r = 1$). V is the voltage applied across the plates. If E is the electric field exists in the capacitor,

then, $$V = E\,d$$

Now a sheet of mica of $\varepsilon_r = 5.4$ is introduced. The thickness of the sheet is 20% of free space distance 'd'. *i.e.*, $0.2\,d$. If E_1 is the electric field intensity in 80% of free space and E_2 is the electric field intensity in mica.

$$V = V_1 + V_2$$

$$E \cdot d = E_1 \times 0.8\,d + E_2 \times 0.2\,d$$

$$E = 0.8\,E_1 + 0.2\,E_2$$

$$E_2 + 4\,E_1 = 5\,E$$

Since D is same, $$E_1 = \frac{D}{\varepsilon_0} \text{ and } E_2 = \frac{D}{\varepsilon_0\,\varepsilon_r}$$

$$\varepsilon_0\,E_1 = \varepsilon_0\,\varepsilon_r\,E_2$$

$$E_1 = 5.4\,E_2$$

$$E_2 + 4\,(5.4\,E_2) = 5\,E$$

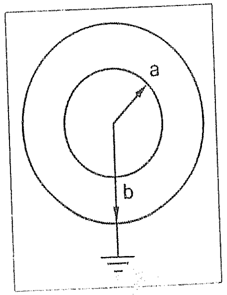
$$E_2 + 21.6\,E_2 = 5\,E$$

$$22.6\,E_2 = 5\,E$$

$$\frac{E}{E_2} = \frac{22.6}{4} = 4.52$$

☺☺

Example 2.28 *The radii of two spheres differ by 4 cm and the capacitance of the spherical condenser is 53.33 p.f. If the outer sphere is earthed, calculate the radii assuming air as dielectric.*



☺ ***Solution:*** The capacitance of spherical condenser is

$$C = 4\pi\,\varepsilon_0 \left(\frac{a\,b}{b-a}\right)$$

$$b - a = 4 \times 10^{-2}\text{ m}$$

$$C = 4\pi \times 8.854 \times 10^{-12}\,\frac{a\,b}{4 \times 10^{-2}}$$

$$= 53.33 \times 10^{-12}$$

$$a\,b = \frac{53.33 \times 10^{-12} \times 4 \times 10^{-2}}{4\pi \times 8.854 \times 10^{-12}}$$

$$a\,b = 0.01917\text{ m}^2$$

$$b - a = 4 \times 10^{-2}\text{ m}$$

$$b = 4 \times 10^{-2} + a$$

Substitute the value of 'b' in ab equation

$$a\,[4 \times 10^{-2} + a] = 0.01917$$

$$a^2 + 4 \times 10^{-2}\,a - 0.01917 = 0$$

$$a = \frac{-4 \times 10^{-2} \pm \sqrt{(4 \times 10^{-2})^2 - 4\,(1 \times -0.01917)}}{2}$$

$$= \mathbf{-0.16 \text{ or } 0.12} \text{ Ans.}$$

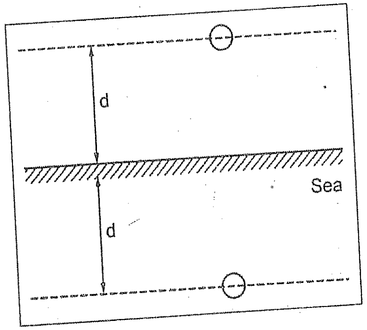
Take the positive value $$a = 0.12\text{ m}$$

$$b = a + 4 \times 10^{-2} = 0.12 + 0.04$$

$$b = \mathbf{0.16\text{ m}} \text{ Ans.}$$

☺☺

Example 2.29 *A wire 8 mm in diameter and 10 km in length is suspended at a constant height of 10 m above water. Calculate the capacitance between conductor and earth.*



☺ ***Solution:***

$$d = 10\text{ m}$$

$$a = \frac{8 \times 10^{-3}}{2} = 4 \times 10^{-3}\text{ m}$$

$$l = 10\text{ km}$$

By method of images, $C = \dfrac{2\pi\,\varepsilon_0}{ln\left[\dfrac{2d-a}{a}\right]}$ $\quad [\because \varepsilon = \varepsilon_0 \text{ and } \varepsilon_r = 1]$

$$C = \frac{2\pi \times 8.854 \times 10^{-12}}{ln\left[\dfrac{20-4\times 10^{-3}}{4\times 10^{-3}}\right]} = 6.53 \text{ p.F/m}$$

Total capacitance between the conductor and earth for 10 km length

$$C = 6.53 \times 10 \times 10^3$$

$$\mathbf{C = 65.3 \text{ nF}} \quad \textbf{Ans.} ☜$$

☺☺

Example 2.30 *Given the potential function $V = 4x + 3y$ in free space, find the energy density.*

☺ ***Solution:***

$$V = 4x + 3y \quad (V)$$

$$E = -\nabla V$$

$$= -\left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z\right)(4x+3y)$$

$$= -4\,\bar{a}_x - 3\,\bar{a}_y \quad (\text{V/m})$$

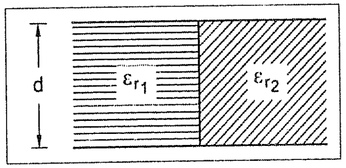
The magnitude of E is $\sqrt{4^2+3^2} = 5$ V/m

$$\text{Energy density} = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\times\frac{10^{-9}}{36\,\pi}\times 25$$

$$= \mathbf{0.11 \times 10^{-9} \text{ J/m}^3} \quad \textbf{Ans.} ☜$$

☺☺

Example 2.31 *Find the capacitance of a parallel plate capacitor containing two dielectrics $\varepsilon_{r_1} = \varepsilon_{r_2} = 3$, each comprising one-half the volume, as shown in Fig. Assume $A = 2\ m^2$ and $d = 1$ mm.*



☺ ***Solution:*** The capacitance of a capacitor consisting of dielectric 1 (ε_{r_1}) alone is C_1.

$$C_1 = \frac{\varepsilon_0\,\varepsilon_{r_1}\,A/2}{d}$$

$$= \frac{8.854\times 10^{-12}\times 2\times 1}{10^{-3}} = 17.708 \text{ nF}$$

The capacitance of a capacitor consisting of dielectric 2 (ε_{r_2}) alone is C_2.

$$C_2 = \frac{\varepsilon_0\,\varepsilon_{r_2}\,A/2}{d}$$

$$= \frac{8.854\times 10^{-12}\times 3\times 1}{10^{-3}} = 26.562 \text{ nF}$$

These two capacitances C_1 and C_2 are in parallel. The effective capacitance is

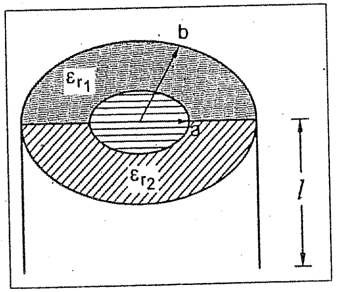
$$C = C_1 + C_2$$

$$= 17.708 + 26.562$$

$$\mathbf{C = 44.270 \text{ nF}}$$

☺☺

Example 2.32 *In the cylindrical capacitor shown in Fig. each dielectric occupies one-half the volume. Find the capacitance. Assume $a = 2$ cm, $b = 4$ cm, $\varepsilon_{r_1} = 2$, $\varepsilon_{r_2} = 3$ and $l = 1$ km.*



☺ ***Solution:*** Capacitance of one-half of the cable

$$C_1 = \frac{\pi\,\varepsilon_0\,\varepsilon_{r_1}\,l}{ln\,(b/a)}$$

$$= \frac{\pi\times 8.854\times 10^{-12}\times 2\times 10^3}{ln\left(\dfrac{4}{2}\right)} = 80.26 \text{ nF}$$

Capacitance of other half of the cable

$$C_2 = \frac{\pi \varepsilon_0 \varepsilon_{r_2} l}{ln\,(b/a)}$$

$$= \frac{\pi \times 8.854 \times 10^{-12} \times 3 \times 10^3}{ln\left(\frac{4}{2}\right)} = 120.39 \text{ nF}$$

These two capacitors are in parallel. The effective capacitance

$$C = C_1 + C_2$$

$$= 80.26 + 120.39 = \mathbf{200.65\ nF}\ \textbf{Ans.}$$ ☜ ☺☺

Example 2.33 *Find the polarization in a dielectric material with $\varepsilon_r = 2.5$ if $D = 5 \times 10^{-8}$ c/m^2.*

☺ ***Solution:***

$$\text{Polarization } P = \chi \varepsilon_0 E$$

$$\text{But } D = \varepsilon_0 \varepsilon_r E$$

$$\text{then } P = \frac{\chi D}{\varepsilon_r}$$

$$\text{But } \chi = \varepsilon_r - 1$$

$$\text{then } P = \left(\frac{\varepsilon_r - 1}{\varepsilon_r}\right) D$$

$$P = \left(\frac{2.5 - 1}{2.5}\right) \times 5 \times 10^{-8}$$

$$= \mathbf{3 \times 10^{-8}\ c/m^2}\ \textbf{Ans.}$$ ☜ ☺☺

Example 2.34 *Determine the electric field strength in a material for which the electric susceptibility is 4 and $P = 2.5 \times 10^{-7}$ c/m^2.*

☺ ***Solution:*** $P = 2.5 \times 10^{-7}$ c/m^2

$$\chi = 4$$

$$E = \frac{P}{\chi \varepsilon_0} = \frac{2.5 \times 10^{-7}}{4 \times 8.854 \times 10^{-12}} = \mathbf{7.06\ kV/m}\ \textbf{Ans.}$$ ☜ ☺☺

Example 2.35 *Determine the capacitance of a capacitor consisting of two parallel plates 30 cm × 30 cm, surface area, separated by 5 mm in air. What is the total energy stored by the capacitor if the capacitors is charged to a PD of 500 V? What is energy density?*

☺ ***Solution:***

Area of plates = 30 cm × 30 cm

= 900 cm^2 = 9×10^{-2} m

Distance of separation = 5 mm = 5×10^{-3} m

Medium = Air

Potential difference = 500 V

From the basic relation of capacitor,

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.854 \times 10^{-12} \times 9 \times 10^{-2}}{5 \times 10^{-3}} \text{ F}$$

$$C = 159.37 \text{ pF}$$

Energy stored in a capacitor, $E = \frac{1}{2} C V^2$ joules

$$E = \frac{1}{2} \times 159.37 \times 10^{-12} \times (500)^2 \text{ J}$$

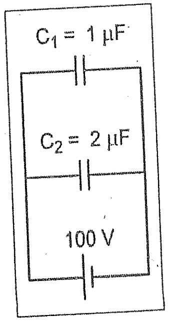
$$E = \mathbf{19.92\ \mu\ Joules}\ \textbf{Ans.}$$ ☜

$$\text{Energy density} = \frac{\text{Energy stored}}{\text{Volume}} \text{ J/m}^3$$

$$= \frac{19.92 \times 10^{-6}}{9 \times 10^{-2} \times 5 \times 10^{-3}} \text{ J/m}^3$$

$$= 0.0442 \text{ J/m}^3$$

∴ The energy density of the capacitor is 0.0442 J/m^3. ☺☺



Example 2.36 *Two capacitors C_1 = 1 μF and C_2 = 2 μF are connected in parallel across a 100 V dc supply. Find the charge on capacitors.*

Given:

C_1 = 1 μF,

C_2 = 2 μF,

V = 100 volts

From the definition of capacitance,

$$C = \frac{q}{V} \Rightarrow q = CV$$

Charge on C_1, $q_1 = C_1 V = 1 \times 10^{-6} \times 100 = 100\ \mu C$

Charge on C_2, $q_2 = C_2 V = 2 \times 10^{-6} \times 100 = 200\ \mu C$

∴ The charges on the capacitors are given as follows:

Capacitor	Charge
$C_1 = 1\ \mu F$	$100\ \mu C$
$C_2 = 2\ \mu F$	$200\ \mu C$

☺☺

Example 2.37 *Two parallel plate capacitors of unknown capacitance when connected in series are known to have an equivalent capacitance of 66.667 μF. When the same two capacitors are connected end-to-end, the equivalent capacitance is found to be 300 μF. What are the two unknown capacitances?*

☺ ***Solution:*** Two capacitors: Parallel $= 300\ \mu F$

$$\text{Series} = 66.67\ \mu F$$

$$C_1 + C_2 = 300\ \mu F$$

From parallel connections, $C_2 = 300\ \mu F - C_1$

$$\frac{C_1 C_2}{C_1 + C_2} = 66.67\ \mu$$

$$\frac{C_1 (300\ \mu - C_1)}{300\ \mu} = 66.67\ \mu$$

$$300\ \mu\, C_1 - C_1^2 = 20000 \times 10^{-12}$$

$$C_1^2 = 300\ \mu\, C_1 - 0.02\ \mu$$

$$C_1^2 - 300\ \mu\, C_1 + 0.02\ \mu = 0$$

$$C_1 = \frac{300\ \mu \pm \sqrt{(0.09 - 0.08)\ \mu}}{2}$$

$$C_1 = \frac{300\ \mu \pm 100\ \mu}{2} = 100\ \mu,\ 200\ \mu$$

∵ In a quadratic equation root $= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2}$

Assuming $\mathbf{C_1 = 100\ \mu F}$ **then** $\mathbf{C_2 = 200\ \mu F}$

$\mathbf{C_1 = 200\ \mu F}$ **then** $\mathbf{C_2 = 100\ \mu F}$ ☺☺

Example 2.38 *A parallel plate capacitor of capacitance 2200 μF is connected across a regulated D.C power supply of 30 V. What is the energy stored in the electric field between the parallel plates in joules?*

☺ ***Solution:*** Capacitance $= 2200\ \mu F$

Voltage $= 30\ V$

Energy stored in a capacitor $= \frac{1}{2} C V^2 = \frac{1}{2} \times 2200\ \mu F \times (30)^2$

$$= 99 \times 10^4\ \mu F \times V^2$$

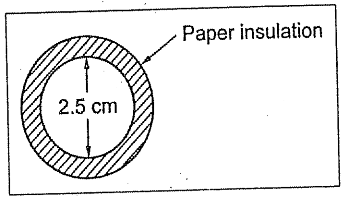
$$= \mathbf{0.99\ J}$$ **Ans.** ☞ ☺☺

Example 2.39 *A coaxial cable has core diameter of 2.5 cm. The outer cable is a thin wire gauge of negligible thickness. The space between the two conductors is impregnated with a paper insulator of thickness 2.13 cm. The dielectric constant of paper is experimentally determined and averaged to 3.5.*

If it is ascertained by the manufacturer that the two conductors never meet, what is the capacitance of the coaxial cable per unit length.

☺ ***Solution:*** A cable with core diameter $= 2.5 \times 10^{-2}$ m

Impregnated paper insulation thickness $= 2.13 \times 10^{-2}$ m



Let, the radius of the core radius $= b = 1.25 \times 10^{-2}$ m

The outer radius of the cable $= a = \dfrac{2.5 + 2.13}{2} = 2.315 \times 10^{-2}$

We know that, $C = \dfrac{2\pi\varepsilon}{ln\,(a/b)}$ Farad/m for a cable

Given, $\varepsilon_r = 3.5$

$$C = \frac{2\pi \times \varepsilon}{ln\,(2.315/1.25)}$$

$= \mathbf{3.159 \times 10^{-10}}$ **Farad/m** **Ans.** ☞ ☺☺

Example 2.40

(a) *Two thin sheet with negligible thickness and dimensions of 5 cm by 10 cm are separated from each other by 0.2 mm throughout their area. Determine the capacitance if dielectric constant of the medium is 1000.*

(b) *If in a capacitor connected across a constant 125 V supply, 2 mJ of energy is stored, determine the value of the capacitance.*

(c) *If the dimensions of the plate in question 'a' are doubled and the distance of separation is doubled then what is the new capacitance?*

☺ ***Solution:***

(a)

$$\varepsilon_r = 1000$$

$$A = 50 \times 10^{-4}\ \text{m}^2$$

$$t = 0.2\ \text{mm} = 0.2 \times 10^{-3}\ \text{m}$$

$$C = \frac{\varepsilon_r \varepsilon_0 A}{t} = \frac{1000 \times 8.854 \times 10^{-12} \times 50 \times 10^{-4}}{0.2 \times 10^{-3}}$$

$$= \mathbf{221.35\ nF}\quad \textbf{Ans.}$$

(b)

$$W = \frac{1}{2} C V^2$$

$$C = \frac{2W}{V^2} = \frac{2 \times 0.002}{(125)^2} = \mathbf{256\ nF}\quad \textbf{Ans.}$$

(c) Assuming square plates of capacitor,

Given,
$$A = 4 \times 10^{-4} \times 50\ \text{cm}^2$$

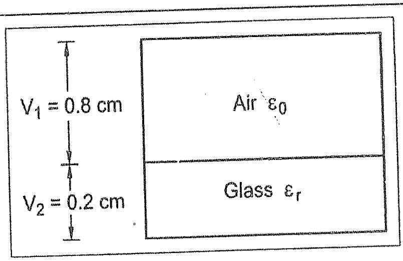
$$t = 0.4 \times 10^{-3}\ \text{m}$$

$$C = \frac{1000 \times 8.854 \times 10^{-12} \times 40 \times 10^{-4} \times 5}{0.4 \times 10^{-3}}$$

$$= \mathbf{442.7\ nF}\quad \textbf{Ans.}$$

☺☺

Example 2.41 *A parallel plate capacitor with a separation $d = 1$ cm has 29 kV applied when free space is the only dielectric. Assume air has a dielectric strength of 30 kV/cm. Show why the air breaks down when a thin piece of glass ($\varepsilon_r = 6.5$) with a dielectric strength of 290 kV/cm and thickness $d_2 = 0.2$ cm is inserted as shown in Fig.*



☺ ***Solution:*** The electric field strength is given as 29 kV/cm. But the dielectric strength of air is 30 kV/cm. Since the electric field strength is less than the dielectric strength, air will not break down.

With the introduction of this glass piece of thickness 0.2 ($\varepsilon_r = 6.5$), the electric field strength changes. Let E_1 and E_2 be the electric field strengths in air and glass piece respectively.

Air: $d_1 = 1 - 0.2 = 0.8$ cm; $d = 1$ cm

Glass: $d_2 = 0.2$ cm

$$E_1 = \frac{D}{\varepsilon_0}$$

$$E_2 = \frac{D}{\varepsilon_0 \varepsilon_r}$$

$$\therefore\ E_2 = \frac{E_1}{\varepsilon_r}$$

$$E_1 = \varepsilon_r E_2 = 6.5\, E_2$$

$$E_1 d_1 + E_2 d_2 = E \cdot d = V$$

$$6.5\, E_2 \times 0.8 + E_2 \times 0.2 = 29\ \text{kV}$$

$$E_2 = \frac{29\ \text{kV}}{5.4} = 5.37\ \text{kV/cm}$$

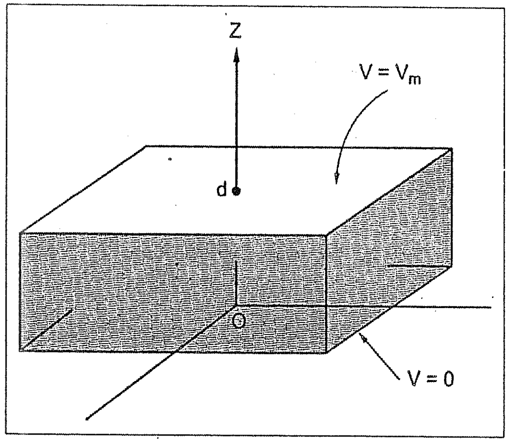
$$E_1 = 6.5\, E_2 = 6.5 \times 5.37 = 34.9\ \text{kV/cm}$$

The new electric field strength 34.9 kV/cm exceeds the dielectric strength of 30 kV/cm of air and hence the air breaks down and will begin to conduct the current.

Example 2.42 *Solve one dimensional Laplace equation to obtain the fiel inside a parallel plate capacitor, and also find the expression for the surfac charge density at two plates.*

☺ ***Solution:*** Consider a parallel plate capacitor as shown in Fig.

$$\text{where,}\quad V = 0 \quad \text{at } z = 0$$
$$V = V_m \quad \text{at } z = d$$



Assuming the region between the plates is charge free, Laplace equation is given by

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(three dimensions)}$$

Since the potential varies only with z, then

$$\frac{\partial^2 V}{\partial z^2} = 0 \quad \text{(one dimension)}$$

$$\text{Integrate}\quad \frac{\partial V}{\partial z} = A$$

$$V = A z + B$$

where, A and B are arbitrary constants.

Apply the boundary conditions,

$$V = 0 \text{ at } z = 0$$
$$B = 0$$

$$V = V_m \text{ at } z = d$$
$$V_m = A \cdot d$$
$$A = \frac{V_m}{d}$$

Substituting these values (constants)

$$V = \frac{V_m}{d} \cdot z \qquad 0 < z < d$$

The electric field intensity E is given by

$$E = -\nabla V$$
$$= -\left[\frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z\right]$$
$$= -\frac{\partial}{\partial z}\left(\frac{V_m}{d} \cdot z\right)\bar{a}_z$$

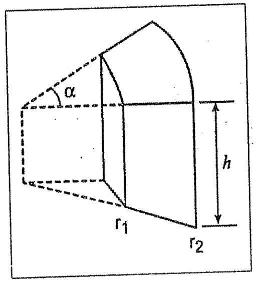
$$\mathbf{E} = \frac{V_m}{d}\bar{a}_z \quad \textbf{Ans.}$$ ☜

Surface charge density $(\rho_s)_{z=0} = [D]_{x=0}\,\bar{a}_z = -\frac{\varepsilon_0 V_m}{d}\bar{a}_z \cdot \bar{a}_z = -\frac{\varepsilon_0 V_m}{d}$

$$(\rho_s)_{z=0} = [D]_{x=d}\,[\bar{a}_z] = \left(-\frac{\varepsilon_0 V_m}{d}\bar{a}_z\right)(-\bar{a}_z)$$
$$= \frac{\varepsilon_0 V_m}{d} \quad \textbf{Ans.}$$ ☜

☺☺

Example 2.43 *Find the capacitance between the inner and outer conductor surfaces shown in Fig. Neglect fringing. Also, find the separation of which results in the same capacitance when the plates are brought into parallel arrangement, with the same dielectric in between.*



The electric flux density between the two plates in cylindrical coordinates is given by,

$$D = D_\phi \bar{a}_\phi$$

The voltage across the plates (one plate at $\phi = 0$ and another plate at $\phi = \alpha$) is given by,

$$V = -\int E \cdot dl$$

$$= -\int_0^\alpha \left(\frac{D_\phi}{\varepsilon} \bar{a}_\phi\right) \cdot (r\, d\phi\, \bar{a}_\phi) \qquad \left[\because E = \frac{D}{\varepsilon}\right]$$

$$= \frac{-D_\phi r}{\varepsilon} \int_0^\alpha d\phi = \frac{-D_\phi r}{\varepsilon} \cdot \alpha$$

$$\therefore D_\phi = \frac{-\varepsilon V}{r\alpha}$$

The charge density on the plate $\phi = \alpha$ is

$$\rho_s = -D_\phi = \frac{\varepsilon V}{r\alpha}$$

The total charge on the plate is then given by

$$Q = \int \rho_s\, ds = \int_0^h \int_{r_1}^{r_2} \frac{\varepsilon V}{r\alpha}\, dr \cdot dz$$

$$= \int_{r_1}^{r_2} \frac{\varepsilon V}{r\alpha}\, h \cdot dr = \frac{\varepsilon V h}{\alpha} \int_{r_1}^{r_2} \frac{1}{r}\, dr$$

$$= \frac{\varepsilon V h}{\alpha}\, ln\, \frac{r_2}{r_1}$$

$$Q = \frac{\varepsilon V h}{\alpha}\, ln\, \frac{r_2}{r_1}$$

$$C = \frac{Q}{V} = \frac{\varepsilon h}{\alpha}\, ln\, \frac{r_2}{r_1}$$ **Ans.** ☜

For the parallel plates, $C = \frac{\varepsilon A}{d}$; $\therefore d = \frac{\varepsilon A}{C}$

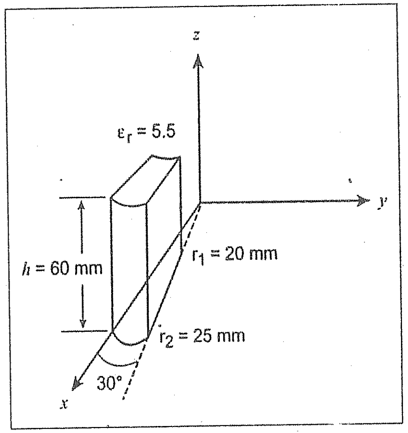
If the area of plate is $A = (r_2 - r_1)\, h$ and capacitance is

$$C = \frac{\varepsilon h}{\alpha}\, ln\, \frac{r_2}{r_1}$$

$$\therefore \text{Separation } d = \frac{\varepsilon h (r_2 - r_1)}{\frac{\varepsilon h}{\alpha}\, ln\left(\frac{r_2}{r_1}\right)}$$

$$\therefore d = \frac{\alpha (r_2 - r_1)}{ln\left(\frac{r_2}{r_1}\right)}$$ ☺☺

Example 2.44 *Find the capacitance between the inner and outer curved conductor surfaces shown in Fig. Neglect fringing. Find the separation which results in the same capacitance when the plates are brought into parallel arrangement, with same dielectric in between.*



$h = 60$ mm $= 0.06$ m ; $\varepsilon_r = 5.5$

$r_1 = 20$ mm $= 0.02$ m

$r_2 = 25$ mm $= 0.025$ m

$\alpha = 30° = \frac{\pi}{6}$

$$C = \frac{\varepsilon h}{\alpha} \ln \frac{r_2}{r_1} = \frac{\varepsilon_0 \varepsilon_r h}{\alpha} \ln \frac{r_2}{r_1}$$

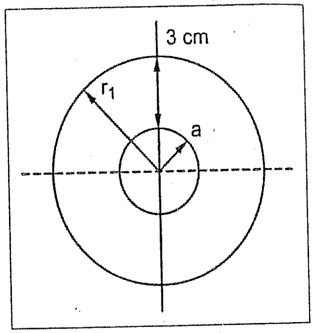
$$= \frac{8.854 \times 10^{-12} \times 5.5 \times 0.06}{\frac{\pi}{6}} \ln \left(\frac{0.025}{0.02}\right)$$

$$C = \mathbf{1.245\ pf}$$ **Ans.** ☜

$$d = \frac{\alpha (r_2 - r_1)}{\ln \frac{r_2}{r_1}} = \frac{\frac{\pi}{6} (0.005)}{\ln \left(\frac{0.025}{0.02}\right)}$$

$$= \mathbf{0.0145\ m\ or\ 1.45\ cm}$$ **Ans.** ☜

Example 2.45 *Obtain the capacitance of a conducting sphere of 2 cm in diameter, which is covered by another medium with ε_r = 2.26 and 3 cm thick.*



☺ ***Solution:*** Let a be the radius of sphere.

$$\text{Radius of sphere} = \frac{d}{2} = 1 \text{ cm}$$

$$r_1 = a + \text{Thickness}$$

$$= 1 + 3 = 4 \text{ cm}$$

Two capacitors are formed in series. Let C_1 be the capacitor due to spherical arrangement of concentric sphere.

$$\therefore \quad C_1 = \left[\frac{4\pi \varepsilon_0 \varepsilon_r}{\frac{1}{a} - \frac{1}{b}}\right]$$

$$= \frac{4\pi \times 2.26 \times 8.865 \times 10^{-12}}{\left[\frac{1}{1 \times 10^{-2}}\right] - \left[\frac{1}{4 \times 10^{-2}}\right]}$$

$$C_1 = 3.3527 \text{ pF}$$

Let C_2 be the capacitor due to isolated sphere.

$$C_2 = 4\pi \varepsilon r_1$$

Between this sphere of radius r_1 and sphere at infinity the dielectric is free space ε_0.

$$C_2 = 4\pi \varepsilon_0 r_1$$

$$= 4\pi \times 8.854 \times 10^{-12} \times 4 \times 10^{-12}$$

$$= 4.4505 \text{ pF}$$

$$C_{eq} = \left[\frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}\right] = \frac{C_1 C_2}{C_1 + C_2}$$

$$= \frac{3.3527 \times 10^{-12} \times 4.4505 \times 10^{-12}}{[3.3527 + 4.4505] \times 10^{-12}}$$

$$C_{eq} = \mathbf{1.9121\ pF}$$ **Ans.** ☜ ☺☺

Example 2.46 *A capacitor consists of squared two metal plates each 100 cm side placed parallel and 2 mm apart. The space between the plates is filled with a dielectric having a relative permittivity of 3.5. A potential drop of 500 V is maintained between the plates. Calculate (i) the capacitance, (ii) the charge of capacitor, (iii) the electric flux density, (iv) the potential gradient.*

☺ ***Solution:***

$$A = 100 \text{ cm} \times 100 \text{ cm} = 1 \text{ m}^2$$

$$\varepsilon_r = 3.5$$

$$d = 2 \times 10^{-3} \text{ m}$$

$$V = 500 \text{ V}$$

(i) Capacitance $C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0 \varepsilon_r A}{d}$

$$= \frac{8.854 \times 10^{-12} \times 3.5 \times 1}{2 \times 10^{-3}} = \mathbf{15.49\ nF}$$ **Ans.** ☜

(ii) Charge of capacitor, $q = CV = 15.49 \times 10^{-9} \times 500$

$$= \mathbf{7.74\ \mu C}$$ **Ans.** ☜

(*iii*) Electric flux density, $D = \varepsilon E$

$$\text{Electric field intensity, } E = \frac{V}{d} = \frac{500}{2 \times 10^{-3}}$$

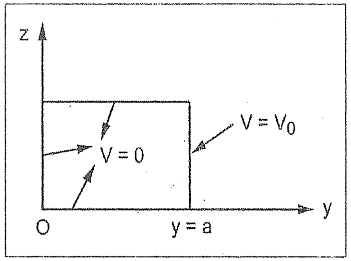
(*iv*) Potential gradient $= 250$ kV/m

$$\therefore D = \varepsilon_0 \varepsilon_r E$$

$$= 8.854 \times 10^{-12} \times 3.5 \times 250 \times 10^3$$

$$= 7.74\ \mu\ c/m^2$$ **Ans.** ☜ ☺☺

Example 2.47 *A rectangular loop is shown in Fig. The voltages at different parts are shown. There exists insulation between $V = 0$ and $V = V_0$ parts. Solve Laplace's equation to find the potential function for the region inside the loop.*



Solution: Potential is varying with respect to y-axis and is independent of x and z-axis.

According to Laplace equation,

$$\nabla^2 V = 0$$

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = 0 \qquad \left[\because \frac{\partial^2 V}{\partial x^2} = 0 \text{ and } \frac{\partial^2 V}{\partial z^2} = 0\right]$$

Integrating,

$$\frac{\partial V}{\partial y} = \int 0 \cdot dy + k_1 = k_1$$

Again integrating,

$$V = \int (k_1)\, dy$$

$$V = k_1 y + k_2 \qquad \ldots (1)$$

At, $y = 0$, $V = 0$

$$\therefore 0 = k_2$$

At $y = a$, $V = V_0$

$$V_0 = k_1 a$$

$$k_1 = \frac{V_0}{a}$$

Substituting in equation (1),

$$V = \frac{V_0 y}{a} \text{ volts}$$ ☺☺

Example 2.48 *Determine the capacitance of concentric cylinders with mixed dielectrics.*

☺ ***Solution:*** Consider a concentric cylinder of inner radius '*a*' and outer radius '*b*'. Two dielectrics occupy in between two coaxial cylinders. Each dielectric (ε_{r1} and ε_{r2}) occupies one-half the volume.

A potential difference V is applied in between two cylinders. The two cylinders are charged at the rate of ρ_l C/m. It is assumed that inner cylinder has charge of ρ_l C/m and outer cylinder has charge of $-\rho_l$ C/m. By applying Gauss's law, the electric field E at any distance r from the axis of one half of cylinder is given by

$$E = \frac{\rho_l}{2\pi \varepsilon_{r1} \varepsilon_0 r}$$

Potential difference between two coaxial cylinder (one half) is

$$V = -\int_b^a E \cdot dr = \frac{-\rho_l}{2\pi \varepsilon_0 \varepsilon_{r1}} \int_b^a \frac{dr}{r}$$

$$= \frac{\rho_l}{2\pi \varepsilon_0 \varepsilon_{r1}}\ ln\,(b/a)$$

Capacitance of one half/unit length is

$$C_1 = \frac{\rho_l}{V} = \frac{\pi \varepsilon_0 \varepsilon_{r1}}{ln\,(b/a)} \text{ F/m}$$

Similarly the electric field E at any distance r from the axis of second half of cylinder is given by,

$$E = \frac{\rho_l}{2\pi \varepsilon_{r2} \varepsilon_0 r}$$

Potential difference between two coaxial cylinders (second half) is

$$V = -\int_b^a \frac{\rho_l}{2\pi \varepsilon_0 \varepsilon_{r2} r} \cdot dr = \frac{\rho_l}{2\pi \varepsilon_0 \varepsilon_{r2}} \ln (b/a)$$

Capacitance of second half/unit length

$$C_2 = \frac{\rho_l}{V} = \frac{\pi \varepsilon_0 \varepsilon_{r2}}{\ln (b/a)} \text{ F/m}$$

Two capacitors are in parallel,

$$C = C_1 + C_2$$

$$C = \frac{\pi \varepsilon_0}{\ln (b/a)} (\varepsilon_{r1} + \varepsilon_{r2}) \text{ F/m} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.49 *A parallel plate capacitor has an area of 0.8 m² separation of 0.1 mm with a dielectric for which $\varepsilon_r = 1000$ and a field of 10^6 V/m. Calculate C and V.*

Given:

$A = 0.8 \text{ m}^2$

$d = 0.1 \text{ mm}$

$\varepsilon_r = 1000$

$E = 10^6 \text{ V/m}$

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d} = \frac{8.854 \times 10^{-12} \times 10^3 \times 0.8}{0.1 \times 10^{-3}}$$

$$= 70.8 \times 10^{-6} \text{ F or } \mathbf{70.8\ \mu F} \quad \textbf{Ans.}$$ ☜

$$V = E \cdot d = 10^6 \times 0.1 \times 10^{-3} = \mathbf{100\ V} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.50 *The relative permittivity ε_r of linear, homogeneous, isotropic dielectric material is 3.6 and the material is covering the space between Z = 0 and Z = 1. If V = – 6000 z volts in the material, find (1) E, (2) P and (3) ρ_v.*

☺ ***Solution: Given:*** $\varepsilon_r = 3.6, \quad Z = 0 \text{ and } Z = 1$

$$V = -6000\, z \text{ volts}$$

$$E = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial x} \vec{a_x} + \frac{\partial V}{\partial y} \vec{a_y} + \frac{\partial V}{\partial z} \vec{a_z} \right]$$

$$= -\frac{\partial(-6000\, z)}{\partial z} \vec{a_z} = \mathbf{-6000\ V/m} \quad \textbf{Ans.}$$ ☜

$$\text{Polarization } P = (\varepsilon_r - 1)\varepsilon_0 E$$

$$= (3.6 - 1) \times 8.854 \times 10^{-12} \times 6000$$

$$= \mathbf{1.38 \times 10^{-7}} \quad \textbf{Ans.}$$ ☜

$$D = \varepsilon_0 E$$

$$= 8.854 \times 10^{-12} \times 6000 = 53.124 \times 10^{-9} \text{ C/m}^2$$

$$\rho_v = \nabla \cdot D$$

$$= \left(\frac{\partial}{\partial x} \vec{a_x} + \frac{\partial}{\partial y} \vec{a_y} + \frac{\partial}{\partial z} \vec{a_z} \right) (53.124 \times 10^{-9})$$

$$\text{Volume charge density } \rho_v = \mathbf{0} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 2.51 *Determine the capacitance of a capacitor consisting of two parallel metal plates 30 cm × 30 cm surface area, separated by 5 mm in air. What is the total energy stored by the capacitor if the capacitor is charged to a potential difference of 500 V? What is the energy density?*

☺ ***Solution:*** ***Data:***

$A = 30 \times 30 \text{ cm}^2 = 0.3 \times 0.3 \text{ m}^2$

$d = 5 \text{ mm} = 5 \times 10^{-3} \text{ m}$

$$C = \frac{\varepsilon_0 A}{d}$$

$$= \frac{8.854 \times 10^{-12} \times 0.09}{5 \times 10^{-3}}$$

$$= 0.159 \times 10^{-9} \text{ F}$$

$$\text{Energy stored} = \frac{1}{2} C V^2 = \frac{1}{2} \times 0.159 \times 10^{-9} \times (500)^2$$

$$= 19.875 \times 10^{-6} \text{ J}$$

$$E = \frac{V}{d} = \frac{500}{5 \times 10^{-3}} = 100 \times 10^{3}\ \text{V/m}$$

$$= \mathbf{100\ kV/m}$$ **Ans.** ☞

$$\text{Energy density} = \frac{1}{2}\,\varepsilon_0\,E^2 = \frac{1}{2} \times 8.854 \times 10^{-12} \times (100 \times 10^{3})^2$$

$$= \mathbf{4.427 \times 10^{-2}\ J/m^3}$$ **Ans.** ☞ ☺☺

Example 2.52 *Determine whether or not the following potential fields satisfy the Laplace's equation.*

(1) $V = x^2 - y^2 + z^2$, *(2)* $V = r\cos\phi + z$, *(3)* $V = r\cos\theta + \phi$.

☺ ***Solution:*** *(1)* $V = x^2 - y^2 + z^2$

Laplace equation in rectangular co-ordinate system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(x^2 - y^2 + z^2) + \frac{\partial^2}{\partial y^2}(x^2 - y^2 + z^2) + \frac{\partial^2}{\partial z^2}(x^2 - y^2 + z^2)$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(2z)$$

$$= 2 - 2 + 2 = 2$$

Since $\nabla^2 V \neq 0$, **it does not satisfy Laplace equation. Ans.** ☞

(2) $V = r\cos\phi + z$

Laplace equation in cylindrical co-ordinate system

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}(r\cos\phi + z) = \cos\phi$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}(r\cos\phi) = \frac{\cos\phi}{r}$$

$$\frac{dV}{d\phi} = \frac{d}{d\phi}(r\cos\phi + z) = -r\sin\phi$$

$$\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{r^2}\frac{\partial}{\partial \phi}(-r\sin\phi)$$

$$= \frac{-1}{r^2}\,r\cos\phi = \frac{-\cos\phi}{r}$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}(r\cos\phi + z) = 1$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}(1) = 0$$

$$\nabla^2 V = \frac{\cos\phi}{r} - \frac{\cos\phi}{r} + 0 = 0$$

It satisfies Laplace equation. Ans. ☞

(3) $V = r\cos\theta + \phi$

Laplace equation in spherical co-ordinate system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta + \phi) = \cos\theta$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cos\theta) = \frac{2r\cos\theta}{r^2}$$

$$= \frac{2\cos\theta}{r}$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta + \phi) = -r\sin\theta$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\,(-r\sin\theta))$$

$$= \frac{1}{r^2\sin\theta}[-2r\sin\theta\cos\theta]$$

$$= \frac{-2\cos\theta}{r}$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}(r\cos\theta + \phi) = 1$$

$$\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$\nabla^2 V = \frac{2\cos\theta}{r} - \frac{2\cos\theta}{r} + 0 = 0$$

It satisfies Laplace's equation. Ans. ☞ ☺☺

Example 2.53 *Solve the Laplace's equation for the potential field in the homogeneous region between the two concentric conducting spheres with radius 'a' and 'b' where $b > a$. $V = 0$ at $r = b$ and $V = V_0$ at $r = a$. Find the capacitance between the two concentric spheres.*

☺ ***Solution:*** Laplace's equation $\nabla^2 V = 0$.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0. \qquad \text{It is not a function of } \theta \text{ and } \phi.$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0$$

$$r^2 \frac{\partial V}{\partial r} = \int 0 + k_1$$

$$\frac{\partial V}{\partial r} = \frac{k_1}{r^2}$$

$$\partial V = \frac{k_1}{r^2} \cdot \partial r$$

$$V = \int \frac{k_1}{r^2}\,\partial r = -\frac{k_1}{r} + k_2$$

Applying boundary conditions,

At $r = b$, $V = 0$

$$0 = -\frac{k_1}{b} + k_2$$

$$k_2 = \frac{k_1}{b}$$

At $r = a$, $V = V_0$

$$V_0 = -\frac{k_1}{a} + k_2$$

Substituting the value of k_2,

$$V_0 = k_1\left[-\frac{1}{a} + \frac{1}{b}\right]$$

$$k_1 = \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}$$

From equation, $$k_2 = \frac{k_1}{b} = \frac{V_0}{b\left(\frac{1}{b} - \frac{1}{a}\right)}$$

$$\therefore V = -\frac{k_1}{r} + k_2$$

$$V = -\frac{V_0}{r\left(\frac{1}{b} - \frac{1}{a}\right)} + \frac{V_0}{b\left(\frac{1}{b} - \frac{1}{a}\right)}$$

Electric field strength, $$E = -\nabla V = -\frac{\partial V}{\partial r}\vec{a_r}$$

$$= -\frac{\partial}{\partial r}\left[\frac{-V_0}{r\left(\frac{1}{b} - \frac{1}{a}\right)}\right]\vec{a_r}$$

$$= \frac{V_0}{\left(\frac{1}{b} - \frac{1}{a}\right)}\left(-\frac{1}{r^2}\right)\vec{a_r} \quad \text{V/m}$$

$$D = \varepsilon E = \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2} \quad \text{c/m}^2$$

Total charge = Flux density × Area

$$= \frac{\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2} \times 4\pi r^2 = \frac{4\pi\varepsilon V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} \cdot C$$

Capacitance $$C = \frac{Q}{V_0} = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} \text{ F}$$ **Ans.** ☞ ☺☺

Example 2.54 *The interface between a dielectric medium having relative permittivity 4 and free space is marked the y = 0 plane. If the electric field next to the interface in the free space region is given by* $E = 5\,\vec{a}_x + 12\,\vec{a}_y + \vec{a}_z$ *V/m, determine E field on the other side of the interface.*

☺ **Solution:** E_1 is the electric field in medium 1.

E_2 is the electric field in medium 2.

$$E_1 = 5\,\vec{a}_x + 12\,\vec{a}_y + \vec{a}_z \quad \text{V/m}$$

The normal direction to $y = 0$ plane is $\vec{a}_y$.

$$E_{N1} = 12\,\vec{a}_y$$

$$E_{tan1} = 5\,\vec{a}_x + \vec{a}_z$$

$$E_1 = E_{N1} + E_{tan1}$$

$$E_{tan1} = E_{tan2} = 5\,\vec{a}_x + \vec{a}_z$$

$$\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}$$

$$E_{N2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} E_{N1} = \frac{1}{4}\,12\,\vec{a}_y = 3\,\vec{a}_y$$

$$\therefore E_2 = E_{N2} + E_{tan2} \doteq 3\,\vec{a}_y + 5\,\vec{a}_x + \vec{a}_z$$

$$\mathbf{E_2 = 5\,\vec{a}_x + 3\,\vec{a}_y + \vec{a}_z}$$ **Ans.** ☞ ☺☺

TWO MARK QUESTIONS AND ANSWERS

1. *Define potential and potential difference.*

Potential difference is defined as the work done in moving a unit positive charge from one point to another point in an electric field.

$$V_{AB} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ V}$$

Potential at any point is defined as the work done in moving a unit positive charge from infinity to that point in an electric field.

$$V = \frac{Q}{4\pi\varepsilon r} \text{ volts}$$

2. *Find the field intensity and potential at a distance 100 mm from a positive point charge of 10 nc.*

$$E = \frac{Q}{4\pi\varepsilon_0 r^2} = \frac{10\times10^{-9}}{4\pi \times \frac{1}{36\pi\times10^9} \times (0.1)^2} = 9 \text{ kV/m}$$

$$V = \frac{Q}{4\pi\varepsilon_0 r} = \frac{10\times10^{-9}}{4\pi \times \frac{1}{36\pi\times10^9} \times (0.1)} = 900 \text{ V}$$

3. *Give the relationship between potential gradient and electric field.*

$$E = -\Delta V$$

$$= -\left(\frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial y}\bar{a}_z\right) \text{ V}$$

4. *Find the electric potential at a point (4, 3)m due to a charge of 10^{-9} C located at the origin in free space.*

$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

$$r = \sqrt{4^2 + 3^2} = 5 \text{ m}$$

$$V = \frac{10^{-9}}{4\pi \times \frac{1}{36\pi\times10^9}\,(5)} = \frac{9}{5} = 1.8 \text{ V}$$

5. *Define dipole and dipole moment.*

Dipole or electric dipole is two equal and opposite point charges are separated by a very small distance.

The product of electric charge and distance (spacing) is known as dipole moment. It is denoted by $m = Q\,l$ coulomb/metre where Q is the charge and l is the distance.

6. ***Write the Poisson's equation and Laplace's equation.***

Poisson's equation $\nabla^2 V = -\frac{\rho}{\varepsilon}$

where, ρ = Volume charge density

ε = Permittivity of the medium

∇^2 = Laplacian operator

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}$$

Laplace's equation $\nabla^2 V = 0$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

7. ***Obtain Poisson's equation from Gauss's law.***

Gauss's law in point form is given by

$$\nabla \cdot D = \rho_v$$

where, D is the electric flux density

ρ_v is the volume charge density

$$\text{But,}\quad D = \varepsilon E$$

$$\varepsilon\, \nabla \cdot E = \rho_v$$

$$\nabla \cdot E = \frac{\rho_v}{\varepsilon}$$

$$\text{But}\quad E = -\nabla V$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

This is Laplace's equation.

8. ***What are the significant physical differences between Poisson's and Laplace's equation?***

Poisson's and Laplace's equations are useful for determining the electrostatic potential V in regions at whose boundaries are known.

When the region of interest contains charges in a known distribution ρ_v Poisson's equation can be used to determine the potential function.

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

When the region is free from charge ($\rho = 0$), Poisson's equation then becomes $\nabla^2 V = 0$ which is Laplace's equation.

9. ***Write the Laplace equation in cylindrical co-ordinate system and mention at least two applications.***

The Laplace equation in cylindrical co-ordinate system is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Applications:

(*i*) To find potential distribution of a coaxial line.

(*ii*) To find potential distribution of parallel infinite long conductor.

10. ***Find the Laplacian of the potential function $V = x^2 + y^2 + z^2$.***

$$V = x^2 + y^2 + z^2$$

$$\frac{\partial V}{\partial x} = 2x \qquad \frac{\partial V}{\partial y} = 2x \qquad \frac{\partial V}{\partial z} = 2x$$

$$\frac{\partial^2 V}{\partial x^2} = 2 \qquad \frac{\partial^2 V}{\partial y^2} = 2 \qquad \frac{\partial^2 V}{\partial z^2} = 2$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 2 + 2 + 2 = 6 \text{ V/m}^2$$

11. ***Define polarization in dielectric material.***

Polarization is defined as dipole moment/unit volume.

$$P = \underset{\Delta v \to 0}{\text{Lt}} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} P_i \quad \text{c/m}^3$$

12. ***What is meant by dielectric breakdown?***

As the electric field applied to dielectric increases sufficiently, the electrons in the dielectric becomes free. Under such large electric field, the dielectric becomes conducting due to presence of large number of free electrons. This condition is called dielectric breakdown.

13. *Define dielectric strength. What is the dielectric strength of air at atmospheric pressure?*

The maximum field intensity that a dielectric can sustain without breakdown.

The dielectric strength of air at atmospheric pressure is 3 mV/m.

14. *Electric flux density in a charge free region is given by $D = 10\,x\,\vec{a_x} + 5\,y\,\vec{a_y} + k\,z^2\,\vec{a_z}$ μc/m², find the constant k.*

$$D = 10\,x\,\vec{a_x} + 5\,y\,\vec{a_y} + k\,z^2\,\vec{a_z}$$

$$\nabla \cdot D = 0 = \frac{\partial}{\partial x}(10\,x) + \frac{\partial}{\partial y}(5\,y) + \frac{\partial}{\partial z}(k\,z^2)$$

$$= 10 + 5 + 2\,k\,z = 0$$

$$2\,k\,z = -15$$

$$k = -7.5\,z$$

15. *A parallel plate capacitor has a charge of 10^{-3} C on each plate while the potential difference between the plates is 1000 V. Calculate the value of capacitance.*

$$Q = 10^{-3}\,C$$

$$V = 1000\,V$$

$$C = \frac{Q}{V} = \frac{10^{-3}}{10^3} = 1 \times 10^{-6}$$

$$C = 1\,\mu F$$

16. *Determine the capacitance of a parallel plate capacitor with two metal plates of size 30 cm × 30 cm separated by 5 mm in air medium.*

$$A = 0.3 \times 0.3 = 0.09\,m^2$$

$$d = 5 \times 10^{-3}\,m$$

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.854 \times 10^{-12} \times 0.09}{5 \times 10^{-3}} = 1.59 \times 10^{-10}\,F$$

$$= 15.9\,nF$$

17. *Write the expression for the capacitance between (a) two parallel plates, (b) two coaxial cylinders.*

(a) $$C = \frac{\varepsilon_0 A}{d}$$

where, A is area of the plates

d is the distance between the plates

ε is the permittivity of the medium

$\varepsilon = \varepsilon_0 \varepsilon_r$

(b) $$C = \frac{\pi \varepsilon_0}{ln\,(d/a)}$$

where, d is the distance between two transmission lines.

a is the radius of cylinders.

18. *Express the value of capacitance for a coaxial cable.*

$$C = \frac{2\pi\,\varepsilon_0\,\varepsilon_r}{ln\,(b/a)}$$

where, b is outer radius

a is inner radius

19. *Find the energy stored in a parallel plate capacitor of 0.5 m by 1 m has a separation of 2 cm and a voltage difference of 10 V.*

$$C = \frac{\varepsilon_0 A}{d}$$

$$= \frac{8.854 \times 10^{-12} \times 0.5 \times 1}{2 \times 10^{-2}} = 2.2135 \times 10^{-10}\,F$$

$$\text{Energy stored } E = \frac{1}{2} C V^2 = \frac{1}{2} \times 2.2135 \times 10^{-10} \times 10^2$$

$$= 1.10675 \times 10^{-8}\text{ Joules}$$

20. *Write the expression for the energy density in electrostatic field.*

$$\frac{\Delta W}{\Delta v} = \frac{1}{2}\varepsilon E^2 = \frac{1}{2} DE$$

21. *Write down the magnetic boundary conditions.*

(a) The normal components of flux density B is continuous across the boundary.

(b) The tangential component of field intensity H is continuous across the boundary.

SUMMARY

Electric potential $V = \frac{Q}{4\pi\varepsilon r}$ volts

Electric field strength $E = -\nabla V$

Electric dipole moment $m = Q\,d$

Poisson's equation $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$

Laplace equation $\nabla^2 V = 0$

Capacitor $C = \frac{Q}{V}$

$$C = \frac{A\,\varepsilon_0\,\varepsilon_r}{d}$$

Capacitance of capacitor having two dielectrics

$$C = \frac{A\,\varepsilon_0\,\varepsilon_{r_1}\,\varepsilon_{r_2}}{d_1\,\varepsilon_{r_2} + (d-d_1)\,\varepsilon_{r_1}} = \frac{A\,\varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{(d-d_1)}{\varepsilon_{r_2}}}$$

Capacitance of capacitor having three dielectrics

$$C = \frac{A\,\varepsilon_0}{\frac{d_1}{\varepsilon_{r_1}} + \frac{d_2}{\varepsilon_{r_2}} + \frac{d_3}{\varepsilon_{r_3}}}$$

Capacitance of isolated sphere $C = 4\pi\varepsilon r$

Particulars	Capacitance
Isolated sphere	$4\pi\varepsilon r$ F
Concentric sphere	$4\pi\varepsilon\left[\frac{a\,b}{b-a}\right]$ F
Coaxial cables	$\frac{2\pi\varepsilon_0\varepsilon_r}{ln\,(b/a)}$ F/m
Transmission lines (Parallel wires)	$\frac{\pi\varepsilon_0}{ln\left(\frac{d}{a}\right)}$ F/m
Infinite single transmission line	$\frac{2\pi\varepsilon}{ln}\left(\frac{2d-a}{a}\right)$ F/m

Electrostatic energy $W = \frac{1}{2}C\,V^2 = \frac{1}{2}Q\,V = \frac{Q^2}{2\,C}$

Energy density $= \frac{1}{2}\varepsilon E^2 = \frac{1}{2}D \cdot E$

Polarization $P = \chi\,\varepsilon_0\,E$

Continuity equation $\oint J \cdot ds = -\frac{dQ}{dt}$ (Integral form)

$\nabla \cdot J = \frac{\partial \rho_v}{\partial t}$ (Differential form)

Conduction current density $J_C = \sigma E$

Displacement current density $J_D = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t}$

Faraday's law $v = -N\frac{d\phi}{dt}$

Self inductance $L = N\frac{d\phi}{di}$

Mutual inductance $M = N_2\frac{d\phi_{12}}{di_1} = N_1\frac{d\phi_{21}}{di_2}$

Coefficient of coupling

$$K = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

$$M = K\sqrt{L_1 L_2}$$

Equivalent inductance

(a) for series connection

$$L = L_1 + L_2 + 2M \quad \text{(aiding)}$$

$$L = L_1 + L_2 - 2M \quad \text{(opposing)}$$

(b) for parallel connection

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \quad \text{(aiding)}$$

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \quad \text{(opposing)}$$

Inductance of solenoid

$$L = \frac{\mu_0 N^2 A}{l}$$

Inductance of toroid

$$L = \frac{\mu_0 N^2 r^2}{2R}$$

Inductance of coaxial cable for unit length

$$L = \frac{\mu_0 I}{2\pi} ln\left(\frac{b}{a}\right)$$

Inductance of coaxial cable with solid inner conductor

$$L = \frac{\mu_0}{4\pi}\left[\frac{1}{2}\mu_r + 2\,ln\left(\frac{b}{a}\right)\right] \quad \text{H/m}$$

Inductance of two transmission lines per unit length

$$L = \frac{\mu_0}{4\pi}\left[\mu_r + 2\,ln\,\frac{d^2}{a\,b}\right]$$

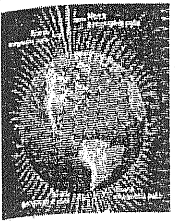
Energy stored in magnetic field

$$W = \frac{1}{2} L I^2$$

Magnetic energy density

$$W = \frac{1}{2} B H$$

□□



3

MAGNETOSTATICS

3.1. MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

Electric charge at rest produces an electric field called electrostatic field. Electric charges in motion that is, electric current produce magnetic field. The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a ***direct current.*** We can ignore the permanent magnet and consider the magnetic field produced by the current element. The magnetic flux lines are enclosed around the element.

Magnetic Flux Density (B)

Magnetic flux density is defined as the magnetic flux passing per unit area. Its unit is Weber/metre^2 or Tesla.

If ϕ webers is total flux passing through an area of A metre^2, then flux density,

$$B = \frac{\phi}{A} \text{ weber/metre}^2$$

This is also defined as $B = \mu H$

where μ is permeability of medium.

$\mu = \mu_0 \mu_r$ (H/m)

$\mu_0 = 4\pi \times 10^{-7}$ (H/m) free space permeability

μ_r is relative permeability of the medium

H is magnetic field intensity (A/m)

Magnetic flux (ϕ)

Magnetic flux is defined as the flux ϕ passing through any area

$$\phi = \int_s B \cdot ds \quad \text{weber}$$

According to the Gauss's law for electric field, the total flux passing through any closed surface is equal to the charge enclosed.

$$\psi = \oint_s D \cdot ds = Q$$

The charge Q is the source of the lines of electric flux, and these lines begin on positive charge and terminate on negative charge.

No such source has found for the lines of magnetic field. The magnetic flux lines are closed and do not terminate on magnetic charge. Hence the Gauss's law for magnetic field is

$$\oint_s B \cdot ds = 0$$

By applying divergence theorem,

$$\iint_S B \cdot ds = \nabla \cdot B$$

$$\text{Then,} \quad \nabla \cdot B = 0$$

The total magnetic flux passing through any closed surface is equal to zero.

3.2. BIOT-SAVART'S LAW

The magnetic flux density produced by a current element at any point in a magnetic field is proportional to the current element and inversely proportional to square of the distance between them.

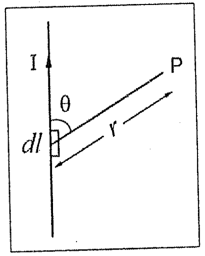


Fig. 3.1. Current element

The magnetic flux density at any point P due to current element I dl is given by

$$dB \propto \frac{I\,dl}{r^2}$$

$$dB = \frac{\mu I\,dl}{4\pi r^2}\,\bar{a}$$

where $\mu = \mu_0 \mu_r$, is permeability of the medium

I dl is the current element

r is the distance between the point P and current element

$\bar{a}$ is the unit vector.

Its magnitude is $$dB = \frac{\mu I\,dl \sin\theta}{4\pi r^2}$$

The magnetic field intensity is given by

$$dH = \frac{I\,dl}{4\pi r^2}\,\bar{a}$$

Its magnitude is $$dH = \frac{I\,dl \sin\theta}{4\pi r^2}$$

This is referred to as ***Ampere's law for current element.***

3.3. MAGNETIC FIELD INTENSITY DUE TO FINITE AND INFINITE CONDUCTOR

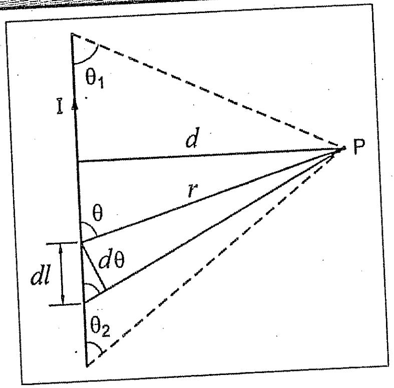


Fig. 3.2.

Consider a conductor of finite length carrying current I and consider a small current element I dl in the conductor at a distance r from the point P where magnetic field is to be determined, as shown in Fig.3.2.

By Biot-Savart's law the flux density at P due to the current element I dl is given by

$$dB = \frac{\mu_0 I\, dl \sin\theta}{4\pi r^2}$$

From the Fig.3.2, $r\, d\theta = dl \sin\theta$ (arc)

Substituting this value in 'dB' equation

$$dB = \frac{\mu_0 I \cdot r\, d\theta}{4\pi r^2} = \frac{\mu_0 I\, d\theta}{4\pi r}$$

From the Fig.3.2, $\frac{d}{r} = \sin\theta$

$$r = \frac{d}{\sin\theta}$$

Substituting the value of r,

$$= \frac{\mu_0 I \sin\theta}{4\pi d}\, d\theta$$

The total flux density is $B = \frac{\mu_0 I}{4\pi d} \int_{\theta_1}^{\pi-\theta_2} \sin\theta\, d\theta$

$$B = \frac{\mu_0 I}{4\pi d} [\cos\theta_1 + \cos\theta_2] \quad \text{weber/m}^2$$

This is the expression for flux density due to finite length conductor.

The magnetic field intensity due to finite length conductor is given by

$$H = \frac{I}{4\pi d} [\cos\theta_1 + \cos\theta_2] \quad \text{A/m}$$

If it is infinitely long $\theta_1 = \theta_2 = 0$, then

$$B = \frac{\mu_0 I}{4\pi d} \cdot 2$$

$$B = \frac{\mu_0 I}{2\pi d}$$

The magnetic flux density at any point due to infinitely long conductor

$$B = \frac{\mu_0 I}{2\pi d} \quad \text{W/m}^2$$

The magnetic field intensity at any point due to infinitely long conductor

$$H = \frac{1}{2\pi d} \quad \text{A/m}$$

3.4. MAGNETIC FIELD INTENSITY ON THE AXIS OF CIRCULAR COIL

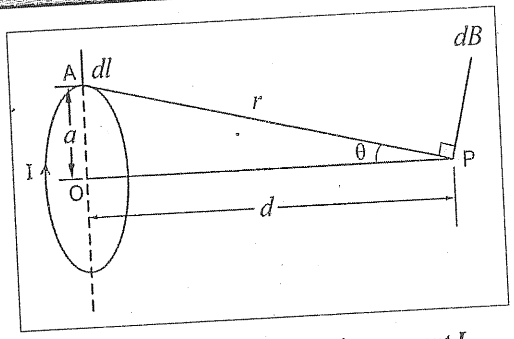


Fig. 3.3. Circular coil carrying current I

Consider a circular coil of radius 'a' carrying a current I and also consider a current element I dl. Let P be any point at a distance d from the centre of the coil as shown in Fig.3.3.

The magnetic flux density at P due to the current element

$$dB = \frac{\mu_0 I\, dl}{4\pi r^2} \sin\theta$$

From ΔAOP, $\sin\theta = \frac{a}{\sqrt{a^2+d^2}}$

$$dB = \frac{\mu_0 I\, dl}{4\pi (a^2+d^2)} \frac{a}{\sqrt{a^2+d^2}}$$

$$dB = \frac{\mu_0 a I\, dl}{4\pi (a^2+d^2)^{3/2}}$$

The magnetic flux density due to circular coil is given by

$$B = \frac{\mu_0 a I}{4\pi (a^2+d^2)^{3/2}} \int dl$$

$$B = \frac{\mu_0 a I}{4\pi (a^2+d^2)^{3/2}} 2\pi a$$

$$B = \frac{\mu_0 a^2 I}{2 (a^2+d^2)^{3/2}} \quad \text{webers/m}^2$$

The magnetic field intensity,

$$H = \frac{I a^2}{2 (a^2+d^2)^{3/2}} \quad \text{A/m}$$

If $d = 0$, the field at the centre,

$$\text{Flux density} \quad B = \frac{\mu_0 a^2 I}{2 a^3}$$

$$B = \frac{\mu_0 I}{2 a} \quad \text{webers/m}^2$$

$$\text{Flux intensity} \quad H = \frac{I}{2 a} \quad \text{A/m}$$

3.5. MAGNETIC FIELD INTENSITY ON THE AXIS OF RECTANGULAR COIL

Consider a rectangular coil ABCD of size $a \times b$ carrying a current of I. Let P be a point at the centre of the coil where magnetic field intensity to be determined. The rectangular coil consists of two finite length of 'a' (AB, CD) and two finite length of 'b' (BC, DA) current carrying conductors.

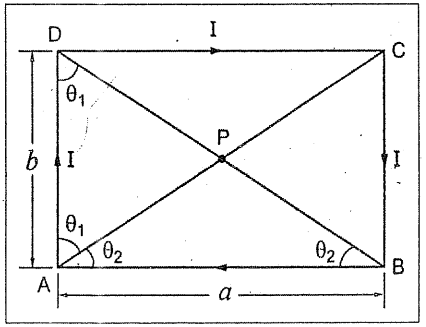


Fig. 3.4.

The magnetic field intensity at the centre P due to segment AB or CD.

$$H = \frac{I}{4\pi d} [\cos\theta_1 + \cos\theta_2]$$

$$\text{But} \quad d = \frac{b}{2}, \quad \theta_1 = \theta_2 = \theta$$

$$\therefore \; H = \frac{2I}{4\pi b} \cdot 2\cos\theta$$

$$= \frac{I}{\pi b} \cos\theta$$

$$\text{where} \quad \cos\theta = \frac{b}{\sqrt{a^2+b^2}}$$

The magnetic field intensity at P due to AB and CD

$$H_1 = \frac{2I}{\pi b} \cos\theta$$

$$H_1 = \frac{2I}{\pi \sqrt{a^2+b^2}}$$

Similarly, the magnetic field intensity at P due to segment BC or DA

$$H = \frac{I}{4\pi d} [\cos\phi_1 + \cos\phi_2]$$

$$\text{But} \quad d = \frac{a}{2}, \quad \phi_1 = \phi_2 = \phi$$

$$\therefore \; H = \frac{2I}{4\pi a} 2\cos\phi$$

$$= \frac{I}{\pi a} \cos\phi$$

$$\text{where} \quad \cos\phi = \frac{a}{\sqrt{a^2+b^2}}$$

The magnetic field intensity at P due to BC and DA

$$H_2 = \frac{2I}{\pi a} \cos\phi$$

$$H_2 = \frac{2I}{\pi \sqrt{a^2+b^2}}$$

The total magnetic field intensity at the centre of rectangular coil is given by

$$H = H_1 + H_2$$

$$= \frac{2\,I}{\pi}\,\frac{2}{\sqrt{a^2+b^2}} = \frac{4\,I}{\pi\,\sqrt{a^2+b^2}} \quad \text{A/m}$$

For square loop of side a,

$$a = b$$

$$H = \frac{4\,I}{\pi\, a\sqrt{2}} = \frac{2\sqrt{2}\,I}{\pi\, a} \quad \text{A/m}$$

3.6. MAGNETIC FIELD INTENSITY ON THE AXIS OF SOLENOID

A solenoid is a cylindrically shaped coil consisting of a large number of turns wound on a non-magnetic frame.

Consider a solenoid which has N turns of length l whose mean radius is 'a' carrying a current I.

Let P be any point on the axis of the solenoid where flux density is to be determined.

Consider an elemental length dl of the solenoid carrying the current of $\frac{N\,I}{l}\cdot dl$ at a distance r from P.

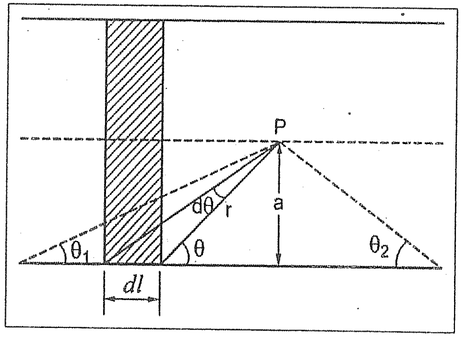


Fig. 3.5. Solenoid

The flux density due to the circular current element at point P is given by

$$dB = \frac{\mu_0\, a^2}{2\, r^3}\,\frac{N\,I}{l}\cdot dl$$

From the Fig., $$\frac{a}{r} = \sin\theta$$

$$r = \frac{a}{\sin\theta}$$

then $$dB = \frac{\mu_0\, a^2}{2\, a^3}\,\sin^3\theta\,\frac{N\,I}{l}\cdot dl$$

$$= \frac{\mu_0 \sin^3\theta}{2\, a}\,\frac{N\,I}{l}\cdot dl$$

From the Fig., $$r\, d\theta = dl\,\sin\theta$$

$$dl = \frac{r\, d\theta}{\sin\theta} \quad \text{and}$$

$$a = r\,\sin\theta$$

Substituting these values in 'dB' equation

$$dB = \frac{\mu_0 \sin^3\theta}{2\, r\,\sin\theta}\,\frac{N\,I}{l}\cdot\frac{r\, d\theta}{\sin\theta}$$

$$= \frac{\mu_0\, N\, I \sin\theta\, d\theta}{2\, l}$$

The total magnetic flux density at point P due to whole solenoid is given by

$$B = \frac{\mu_0\, N\, I}{2\, l}\int_{\theta_1}^{\pi-\theta_2} \sin\theta\, d\theta$$

$$B = \frac{\mu_0\, N\, I}{2\, l}\,[\,\cos\theta_1 + \cos\theta_2\,]$$

The magnetic field intensity due to solenoid at P is given by

$$H = \frac{N\,I}{2\, l}\,[\,\cos\theta_1 + \cos\theta_2\,]$$

Case 1: If the point P is on the middle of the axis of solenoid, $\theta_1 = \pi - \theta_2$

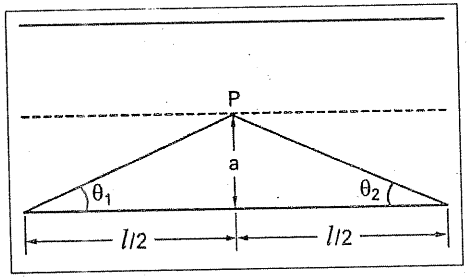


Fig. 3.6.

The magnetic flux density at mid-point P on the axis of solenoid is

$$B = \frac{\mu_0 N I}{2l} \, 2\cos\theta_1$$

$$= \frac{\mu_0 N I}{l} \cos\theta_1$$

where $\cos\theta_1 = \dfrac{l/2}{\sqrt{(l/2)^2 + a^2}}$

$$\therefore \; B = \frac{\mu_0 N I}{2\sqrt{(l/2)^2 + a^2}}$$

The magnetic field intensity is given by

$$H = \frac{N I}{2\sqrt{(l/2)^2 + a^2}}$$

Case 2: If the point P at the end of the axis of solenoid, $\theta_2 = 90°$.

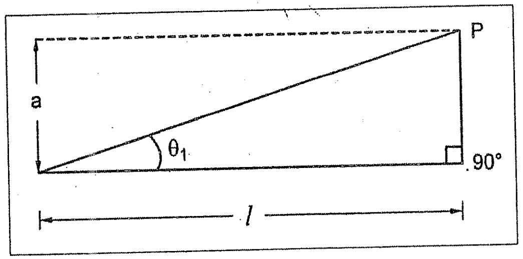


Fig. 3.7.

The magnetic flux density at end point P on the axis of solenoid is given by

$$B = \frac{\mu_0 N I}{2l} \cos\theta_1$$

where $\cos\theta_1 = \dfrac{1}{\sqrt{(l^2 + a^2)}}$

$$B = \frac{\mu_0 N I}{2\sqrt{l^2 + a^2}}$$

The magnetic field intensity due to solenoid is given by

$$H = \frac{N I}{2\sqrt{l^2 + a^2}}$$

3.7. MAGNETIC FIELD INTENSITY AT THE CENTRE OF THE TOROIDAL COIL

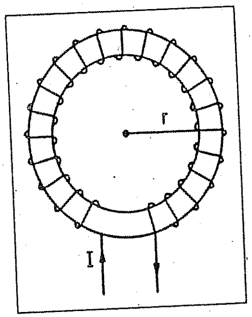


Fig. 3.8.

Consider a toroidal coil of mean radius r having current I and N number of turns.

According to Ampere's circuital law, the line integral of H around any closed path is equal to the current.

$$\oint H \cdot dl = N I$$

$$H \int_0^{2\pi r} dl = N I$$

$$H \, 2\pi r = N I$$

The magnetic field $H = \dfrac{N I}{2\pi r}$

The magnetic flux density is $B = \mu_0 H = \dfrac{\mu_0 N I}{2\pi r}$

$$B = \frac{\mu_0 N I}{2\pi r}$$

3.8. AMPERE'S CIRCUITAL LAW AND APPLICATIONS

Ampere's circuital law states that the line integral of magnetic field intensity H about any closed path is exactly equal to the direct current enclosed by that path.

$$\oint H \cdot dl = I$$

Ampere's law is a special case of Biot-Savart's law. Ampere's law is similar to Gauss's law and it is easily applied to determine magnetic field intensity H when the current distribution is symmetrical.

Applications

Applications of Ampere's law to an infinite line current and infinite line current sheet and infinitely long coaxial transmission line are discussed as follows.

3.8.1. Infinite Line Current

Consider an infinitely long filamentary current I along the Z-axis as shown in Fig.3.9. Let P be any point at which H has to be determined. Consider a closed path passing through P which is known as Amperian path. Since this path encloses the whole current I, according to Ampere's law.

$$I = \oint H \cdot dl = H(2\pi r)$$

$$H = \frac{1}{2\pi r} \bar{a}_r$$

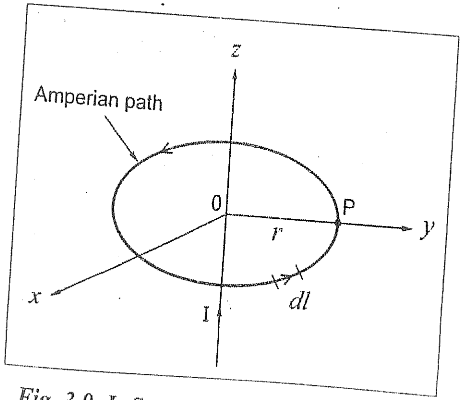


Fig. 3.9. Infinite filamentary line current

Infinite Sheet of Current

Consider an infinite current sheet in the $z = 0$ as shown in Fig.3.10. Let the current density of sheet is $K = k_y \bar{a}_y$ A/m. Consider a rectangular (Amperian path) closed path by applying Ampere's law.

$$\oint H \cdot dl = I = k_y\, b$$

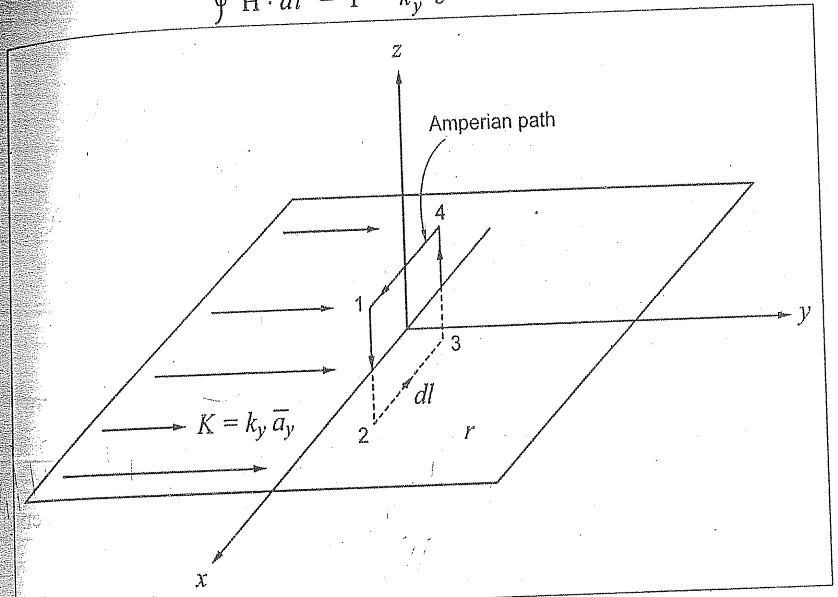


Fig. 3.10. Infinite sheet

Due to the infinite sheet, the sheet can be considered as consisting of filamentary pairs so that the characteristics of H for a pair are the same for the infinite sheet of current.

$$H = \begin{cases} H_0\, \bar{a}_x & z > 0 \\ -H_0\, \bar{a}_x & z < 0 \end{cases}$$

H on one side of the sheet is the negative of that on the other side.

The line integral of rectangular path is given by

$$\oint H \cdot dl = \int_1^2 H \cdot dl + \int_2^3 H \cdot dl + \int_3^4 H \cdot dl + \int_4^1 H \cdot dl$$

$$= 0(-a) + (-H_0)(-b) + 0(a) + H_0\,(b)$$

$$\oint H \cdot dl = 2 H_0 \, b$$

Equating $\oint H \cdot dl$ equations

$$2 H_0 \, b = k_y \, b$$

$$H_0 = \frac{k_y}{2}$$

Then,

$$H = \frac{k_y}{2} \bar{a}_x \qquad z > 0$$

$$= \frac{-k_y}{2} \bar{a}_x \qquad z < 0$$

In general, for an infinite sheet of current density K A/m

$$H = \frac{1}{2} K \times \bar{a}_n$$

where a_n is the unit normal vector.

3.8.3. Infinitely Long Coaxial Transmission Line

Consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis as shown in Fig.3.11.

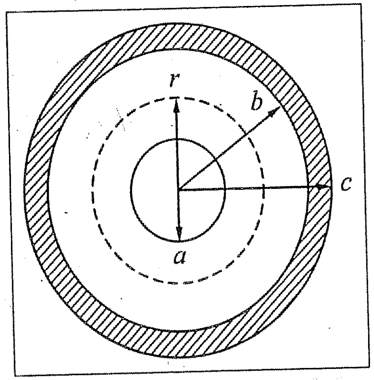


Fig. 3.11. Cable

Consider a cable with a solid inner conductor of radius 'a' carries current I and the outer annular of inner radius 'b' and outer radius 'c' carries – I.

(a) ***The field intensity within the solid inner conductor of radius (0 ≤ r ≤ a)***

According to Ampere's law, the magnetic field around a closed path equal to the current enclosed.

$$\int H \cdot dl = \frac{I}{\pi a^2} \pi r^2$$

$$H \cdot 2\pi r = I \frac{r^2}{a^2}$$

$$H = \frac{I r}{2\pi a^2} \quad \text{A/m} \qquad 0 < r < a$$

(b) ***The field intensity in between inner and outer conductor (a ≤ r ≤ b)***

$$\oint H \cdot dl = I$$

$$\therefore \; H = \frac{I}{2\pi r} \quad \text{A/m} \qquad a \le r \le b$$

(c) ***The field intensity in the annular (outer conductor) (b < r < c).***

$$\oint H_\phi \cdot dl = I + \int_b^r \int_0^{2\pi} \frac{-I}{\pi (c^2 - b^2)} r \cdot d\phi \cdot dr$$

$$= I - \left[\frac{1}{\pi (c^2 - b^2)} \int_0^{2\pi} d\phi \, \frac{r^2}{2} \right]_b^r$$

$$= I - \left[\frac{I (r^2 - b^2)}{2\pi (c^2 - b^2)} \phi \right]_0^{2\pi}$$

$$= I - \frac{I (r^2 - b^2)}{(c^2 - b^2)}$$

But $\oint H_\phi \cdot dl = H_\phi 2\pi r$

$$H_\phi 2\pi r = I \left[1 - \frac{(r^2 - b^2)}{(c^2 - b^2)} \right]$$

$$H_\phi = \frac{I}{2\pi r} \left[1 - \frac{(r^2 - b^2)}{(c^2 - b^2)} \right] \qquad b < r < c$$

(d) The field intensity outside the cable $r \geq c$

$$H_\phi = \frac{I}{2\pi r}[1-1] \qquad [\because r = c]$$

$$= 0$$

or

Since no current is enclosed.

$$\int_l H_\phi \, dl = 0$$

$$\therefore \; H_\phi = 0$$

3.9. LORENTZ FORCE EQUATION AND APPLICATION

In a electric field, the force acting on a charged particle is given by

$$F_E = Q E$$

This electric force is directly proportional to both electric field intensity (E) and magnitude of the charge (Q) in the same direction as the electric field intensity.

In a magnetic field of flux density (B), a charged particle moving with velocity 'v' experiences a force which is given by

$$F_M = Q \, v \times B$$

The magnetic force is proportional to the product of the magnitudes of the charge, its velocity and the flux density B and to the sine of the angle between the vectors v and B. The direction of the force is perpendicular to both v and B.

The force on a moving particle due to combined electric and magnetic fields is given by

$$F = F_E + F_M = Q E + Q \, v \times B$$

$$= Q(E + v \times B)$$

This equation is called as ***Lorentz force equation.***

3.9.1. Force on a Differential Current Element

When a charged particle moving with velocity v in a steady magnetic field, there is a force acting on this particle.

Consider a differential element of charge consists of a large number of very small charges moving with velocity v in a magnetic field of flux density B. The differential force exerted on a differential element of charge is the sum of the forces on the individual charges.

$$dF = dQ \; v \times B$$

The differential element of charge is expressed in terms of volume charge density ρ_v.

$$dQ = \rho_v \cdot dv$$

$$\text{Then,} \quad dF = \rho_v \, dv \; v \times B$$

$$\text{But current density} \quad J = \rho_v \, v$$

Substituting the value of J in differential force

$$dF = J \times B \, dv$$

J dv can be expressed as differential element I dl.

Lorentz force equation is applied to the differential current element.

$$\therefore \quad dF = I \, dl \times B$$

The force due to entire conductor,

$$F = \oint F = \oint I \, dl \times B$$

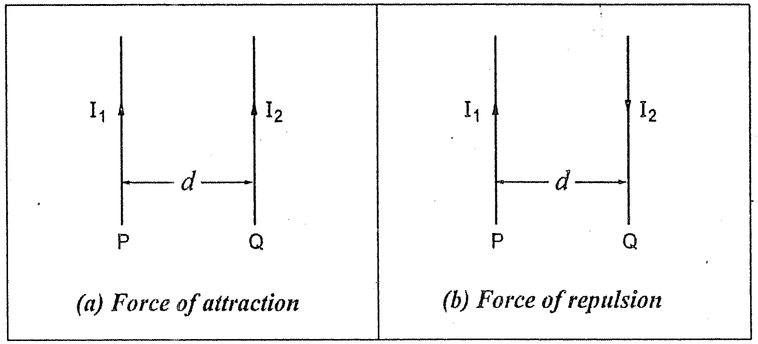
$$= I \, l \times B$$

The magnitude of the force is $F = B I l \sin\theta$

where, θ is the angle between B and I.

3.10. FORCE BETWEEN THE CURRENT CARRYING CONDUCTORS

Consider two straight, long parallel conductors P and Q separated by a distance 'd'. Let I_1 and I_2 be the currents flowing in conductors P and Q respectively.



(a) Force of attraction (b) Force of repulsion

Fig. 3.12. Parallel conductors

Consider a conductor P produces a magnetic field whose flux density is B at conductor Q.

$$B = \frac{\mu_0 I_1}{2\pi d}$$

The force on conductor Q due to P,

$$F = B I_2 l$$

where, l is the length of the conductor.

$$F = \frac{\mu_0 I_1}{2\pi d} I_2 l$$

If currents of conductors P and Q are flowing in same direction, there is a force of attraction.

If currents are flowing in opposite direction, there is a force of repulsion. However, the magnitude of force is same.

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi d} \text{ N}$$

If the conductors are infinitely long, the force per unit length is

$$F/l = \frac{\mu_0 I_1 I_2}{2\pi d} \text{ N/m}$$

3.11. MAGNETIC TORQUE AND MAGNETIC MOMENT

When a current loop is placed parallel to a magnetic field, forces act on the loop that tend to rotate it. The tangential force multiplied by the radial distance at which it acts is called ***Torque*** or ***Mechanical moments*** on the loop.

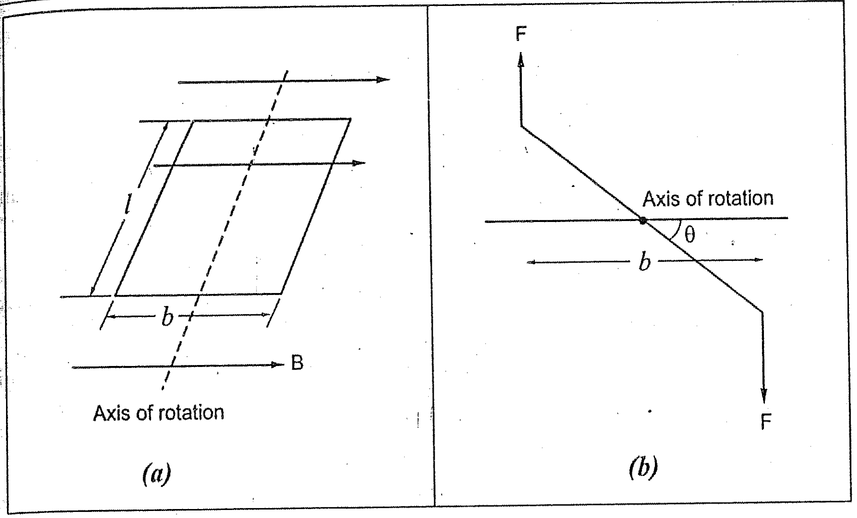


Fig. 3.13. Current loop

Consider the rectangular loop of length 'l' and breadth 'b' carrying a current I in a uniform magnetic field of flux density B as shown in Fig.3.13. The loop makes an angle θ with respect to magnetic flux density B.

The force acting on the loop

$$F = B I l \sin\theta$$

If the loop plane is parallel to the magnetic field, the total torque on the loop,

$$\begin{aligned} T &= 2 \times \text{Torque on each side} \\ &= 2 \times \text{Force} \times \text{Distance} \\ &= 2 B I l \sin\theta \frac{b}{2} = B I l b \sin\theta \\ &= B I A \sin\theta \qquad (\text{Area } A = l\, b) \end{aligned}$$

Torque is given by, $T = B I A \sin\theta$

The ***magnetic moment*** of loop is I A.

The magnetic moment is a vector with the direction given by the unit normal $\hat{n}$ to the plane of the loop

$$m = I A \hat{n}$$

$$T = m B \sin\theta \ \hat{n}$$

In vector form, torque can be represented as

$$T = m \times B$$

$$m = \frac{T}{B}$$

The ***magnetic moment*** is defined as the maximum torque on loop per unit magnetic induction.

3.12. MAGNETIC POTENTIAL

Scalar Magnetic Potential

Ampere's law stated that the line integral of the field H around a closed path is equal to the current enclosed.

$$\oint H \cdot dl = I$$

If no current is enclosed *i.e.,* $J = 0$.

$$\oint H \cdot dl = 0$$

Magnetic field H can be expressed as negative gradient of a scalar function.

$$H = -\nabla V_m$$

where, V_m is called scalar magnetic potential.

$$V_m = -\int H \cdot dl$$

This scalar potential also satisfies Laplace's equation.

$$\text{In free space,} \quad \nabla \cdot B = 0$$

$$\mu_0 \nabla \cdot H = 0$$

$$\text{But,} \quad H = -\nabla V_m$$

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

$$-\mu_0 \nabla^2 V_m = 0$$

$$\nabla^2 V_m = 0$$

Vector Magnetic Potential

Scalar magnetic potential exists if there is no current enclosed *i.e.,* $\oint H \cdot dl = 0$. If current is enclosed, the potential which depends upon current element (vector quantity) is no more scalar but it is vector quantity.

Since the divergence of a vector is a scalar, vector potential is expressed in curl.

$$\text{i.e.,} \quad \nabla \cdot B = 0$$

$$B = \nabla \times A$$

where, A is magnetic vector potential.

$$\text{Take curl on both sides,} \quad \nabla \times B = \nabla \times \nabla \times A$$

$$\text{By the identity,} \quad \nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

$$\text{But} \quad \nabla \times B = \mu J$$

$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu J$$

$$\text{For the steady dc,} \quad (\nabla \cdot A) = 0$$

$$\text{then,} \quad -\nabla^2 A = \mu J$$

$$\bar{a}_x \nabla^2 A_x + \bar{a}_y \nabla^2 A_y + \bar{a}_z \nabla^2 A_z = -\mu(\bar{a}_x J_x + \bar{a}_y J_y + \bar{a}_z J_z)$$

$$\text{Equating} \quad \nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

They are in the form of Poisson's equation. From the above equations, the magnetic vector potential can be written as

$$A_x = \frac{\mu}{4\pi} \int_v \left(\frac{J_x}{r}\right) dv$$

$$A_y = \frac{\mu}{4\pi} \int_v \left(\frac{J_y}{r}\right) dv$$

$$A_z = \frac{\mu}{4\pi} \int_v \left(\frac{J_z}{r}\right) dv$$

The general, magnetic vector potential can be expressed as

$$A = \frac{\mu}{4\pi} \iiint \frac{J}{r} \, dv$$

3.13. MAGNETIC MATERIALS

Magnetic field is present around a current carrying conductor. It is also exist around magnet. An electron revolving around the nucleus of its atom in its orbit forms a tiny electric current loop. This current loop produces a magnetic field. Most of the substances would exhibit magnetic effects and some of them are magnetically weak. Depending on their magnetic behaviour, substances can be classified into three groups. They are:

(*i*) Diamagnetic
(*ii*) Paramagnetic
(*iii*) Ferromagnetic

The metals and other elements having slight magnetic properties are called ***diamagnetic*** materials in which the magnetization is opposite to the applied field. If the magnetization is in the same direction as the applied field, such materials are called as ***paramagnetic*** materials. In fact, these two materials which have feeble magnetic effects are called non-magnetic materials. Ferro magnetic materials show very strong magnetic effects. The magnetization is in the same direction as the field. Iron, steel, Nickel and Cobalt are known as ferromagnetic materials.

Relative permeability decides the material which belongs to whether diamagnetic, paramagnetic or ferromagnetic. For vacuum, $\mu_r = 1$, vacuum is taken as reference medium.

If $\mu_r \leq 1$, the materials are diamagnetic.
If $\mu_r \geq 1$, the materials are paramagnetic.
If $\mu_r >> 1$, the materials are ferromagnetic.

Substances classified according to their relative permeability are tabulated in Table 3.1.

Table 3.1. Classification of substances according to their relative permeability

Substance	Magnetic type	Relative permeability (μ_r)
Silver	Diamagnetic	0.99998
Copper	Diamagnetic	0.999991
Vacuum	Non-magnetic	1
Aluminium	Paramagnetic	1.00002
Palladium	Paramagnetic	1.0008
Cobalt	Ferromagnetic	250
Nickel	Ferromagnetic	600

3.13.1. Magnetization

A bar magnet is composed of a large number of a magnetic dipoles as in Fig.3.15. As in the case of an electric dipole consists of equal and opposite charges with very small separation, a small bar magnet with pole strength Q_m and length l may be treated as ***magnetic dipole*** whose ***magnetic moment*** is $Q_m\, l$.

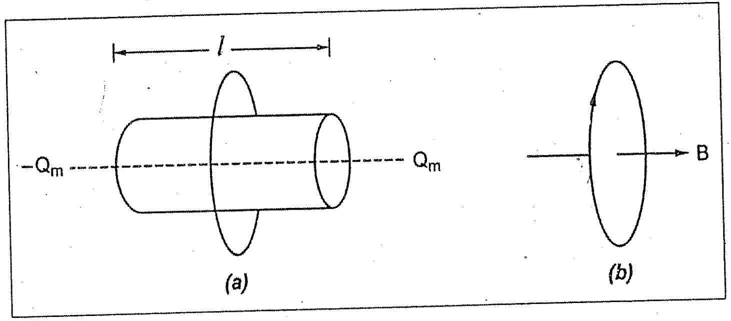


Fig. 3.14. Dipole moment and equivalent current loop

A magnetic dipole is considered as equivalent to current loop with area A and carrying a current I as shown in Fig.3.14. This loop has a magnetic moment

$$m = I A$$

$$\therefore \; Q_m l = I A$$

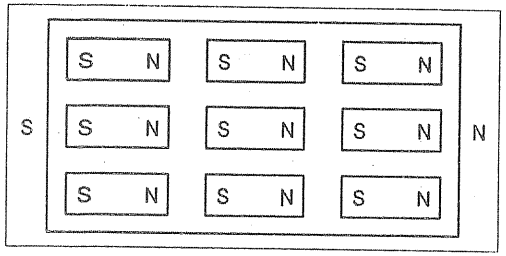


Fig. 3.15. Bar magnet composed of magnetic dipoles

If A is the area of cross-section of the bar magnet and l its axial length, the volume of the magnet is A l (Fig.3.15). The dipole moment of the bar magnet is $Q_m l$.

The net dipole moment per unit volume is called ***Magnetization.*** It is represented as vector M.

$$M = \frac{\text{Dipole moment}}{\text{Volume}} = \frac{Q_m l}{\text{Volume}}$$

$$M = \frac{Q_m}{A} \bar{a} \quad \text{A/m}$$

M can be defined at any point in the bar magnet by

$$M = \underset{\Delta v \to 0}{\text{Lt}} \frac{m}{\Delta v} \quad \text{A/m} \qquad [\because m = I\,A = Q_m l$$

Magnetic Susceptibility

Magnetic susceptibility is defined as the ratio of magnetization to the magnetic field intensity. It is denoted by χ_m.

$$\chi_m = \frac{M}{H}$$

It is dimensionless quantity.

In a free space, the flux density is given by

$$B = \mu_0 H$$

For magnet, the flux density is given by

$$B = \mu_0 (H + M)$$

$$= \mu_0 H \left(1 + \frac{M}{H}\right)$$

$$B = \mu_0 H (1 + \chi_m)$$

$$\text{But } B = \mu H$$

Equating these two equations,

$$\mu = \mu_0 (1 + \chi_m)$$

$$\text{But } \mu = \mu_0 \mu_r$$

From these two equations, the relative permeability is

$$\mu_r = 1 + \chi_m$$

3.13.2. Magnetization (B-H) Curve

The ratio between B and H may be constant for all values of H, especially in the case of ferromagnetic materials. The characteristics curve showing the variation of flux density (B) with field intensity (H) is called ***Magnetization Curve.***

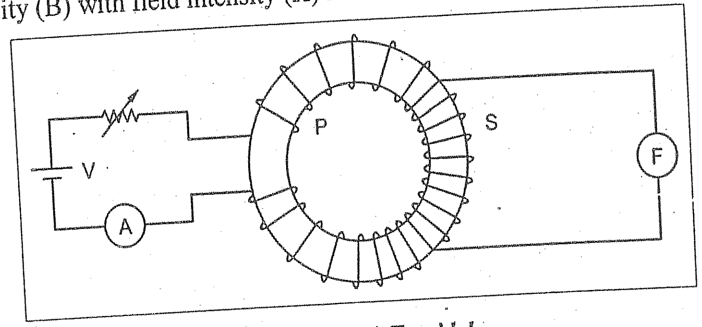


Fig. 3.16. (a) Toroidal

Consider a toroid with ferromagnetic core which has primary and secondary coils. The primary coil is excited by a variable DC power supply which produces the change in field intensity H and its corresponding effect can be measured by a flux meter at the secondary coil as shown in Fig.3.16(a). The value of H can be increased or decreased by increasing or decreasing the current through the toroid. Now H is increased from zero to a certain maximum value and the corresponding value of B is noted. This is represented as ***oc*** in B-H curve as shown in Fig.3.16(*b*). The point ***c*** is called magnetic saturation.

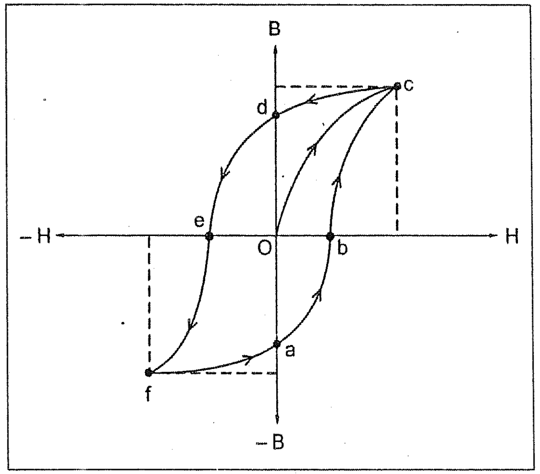


Fig. 3.16. (b) Toroid and its hysteresis loop

If H is reduced to zero gradually, B will not decrease to zero but has a value ***od***. This is called ***residual*** or ***remanent*** value. In order to bring down the value of B to zero (*i.e.*, to demagnetize the toroid), H has to be applied in reverse (negative) direction. When H is reversed, then B is reduced to zero at point ***e***. The field at ***e*** (H = – H_e) is called ***coercive force***. The value of H is further increased in the negative direction, the value of B is reached in saturation level in negative direction at ***f***. By taking H back from its value for negative saturation to its value for positive saturation, a similar curve ***fabc*** is obtained. It is concluded that B always lags behind H. This is called ***hysteresis***. The closed loop ***cdefabc*** is termed as ***BH loop*** or ***hysteresis loop***.

3.14. MAGNETIC BOUNDARY CONDITIONS

The boundary conditions between the two different magnetic materials are given below.

1. The tangential component of magnetic field intensity is continuous across the boundary.

2. The normal component of magnetic flux density is continuous across the boundary.

Consider a boundary between two isotropic homogeneous media with permeabilities μ_1 and μ_2 as shown in Fig.3.17.

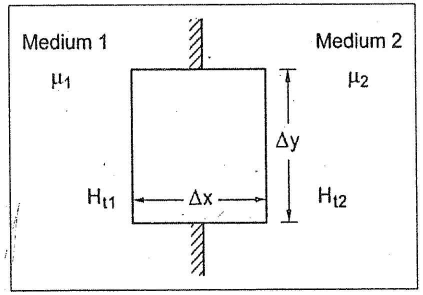


Fig. 3.17. Boundary between two magnetic media

Consider a small rectangle of width Δx and length Δy at the boundary of the two media. Let H_{t_1} be the tangential component of magnetic field in medium 1 and H_{t_2} be the tangential component of magnetic field in medium 2.

According to Ampere's law

$$\oint H \cdot dl = I$$

If there is no current enclosed by the path

$$\oint H \cdot dl = 0$$

$$H_{t_1} \Delta y - H_{t_2} \Delta y = 0$$

$$H_{t_1} = H_{t_2}$$

The tangential component of H in medium 1 is same as in medium 2 *i.e.*, the tangential component of H is continuous.

Consider a pill box of surface area *ds* across the boundary between two isotropic homogeneous media, as shown in Fig.3.18.

3.15. INDUCTANCE

Professor Oersted demonstrated that magnetic field is produced by changing electric current in 1820. Michael Faraday showed that an electric current is induced by changing the magnetic field in 1831. This phenomena is known as ***Electromagnetic Induction.***

3.15.1. Self Inductance

By Faraday's law, changing current will produce an induced emf in the circuit to oppose the change in flux. This phenomena is known as self-induction. ***Self-inductance*** of a circuit is the property of the circuit by which changing current induces emf in the circuit to oppose the changing current.

Consider a coil having 'N' number of turns. If changing current (alternating current) is applied, the emf is induced in the coil.

The induced emf is proportional to the rate of change of current.

$$v \propto \frac{di}{dt}$$

$$v = L\frac{di}{dt}$$

where L is the self inductance (*i.e.*, constant of proportionality).

From Faraday's law, the induced emf is

$$v = N\frac{d\phi}{dt}$$

Equating these two equations,

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$

$$L = N\frac{d\phi}{dt}$$

If the permeability is constant, then

$$L = \frac{N\phi}{i}$$

The inductance is defined as the ratio of total magnetic flux linkage to the current through the coil.

3.15.2. Mutual Inductance

Consider two coils 1 and 2 magnetically coupled together as shown in Fig.3.20. The changing i_1 produces a flux ϕ_1. If a second coil is placed near the first coil, some of the flux links coil 2, say ϕ_{12}. The induced emf in coil 2 is given by

$$v_2 = N_2\frac{d\phi_{12}}{dt}$$

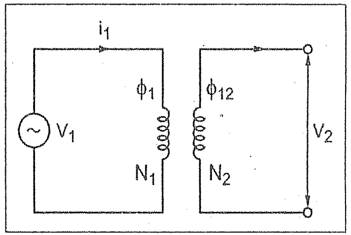


Fig. 3.20. Coupled circuit

Since flux ϕ_{12} is produced by first coil current i_1, the induced emf v_2 in coil 2 is proportional to the rate of change of current i_1.

$$v_2 \propto \frac{di_1}{dt}$$

$$v_2 = M\frac{di_1}{dt}$$

where, M is the mutual inductance between the two coils.

Equating these two equations,

$$M\frac{di_1}{dt} = N_2\frac{d\phi_{12}}{dt}$$

$$M = N_2\frac{d\phi_{12}}{di_1}$$

If the permeability is constant,

$$M = \frac{N_2\,\phi_{12}}{i_1}$$

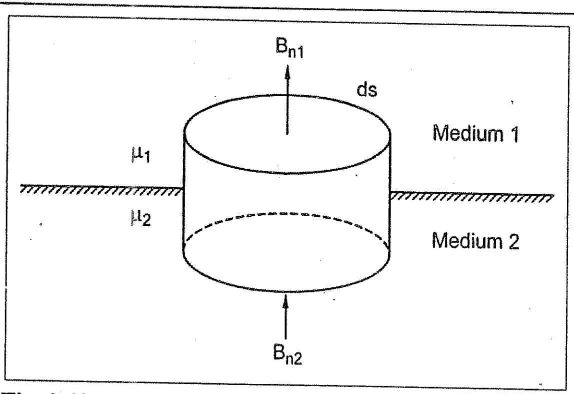


Fig. 3.18. Pill box at the boundary of two magnetic media

Let B_{n_1} be the normal component of magnitude flux density in medium 1 and B_{n_2} be the normal component of magnetic flux density in medium 2.

By Gauss's law for magnetic field

$$\iint_S B \cdot ds = 0$$

$$B_{n_1}\, ds - B_{n_2}\, ds = 0$$

$$B_{n_1} = B_{n_2}$$

The normal component of B is continuous across the boundary.

Consider the magnetic lines away from the normal across the boundary as shown in Fig.3.19.

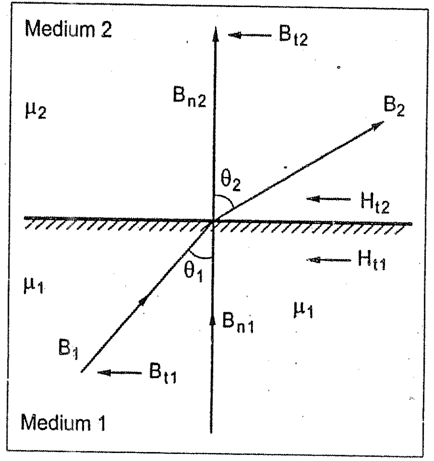


Fig. 3.19. Boundary surface between two magnetic media

From the Fig.3.19,

$$B_{n_1} = B_1 \cos\theta_1$$

$$B_{n_2} = B_2 \cos\theta_2$$

$$\text{But} \quad B_{n_1} = B_{n_2} \qquad \text{(boundary condition)}$$

$$B_1 \cos\theta_1 = B_2 \cos\theta_2 \qquad \ldots (1)$$

$$H_{t_1} = \frac{B_{t_1}}{\mu_1}$$

$$H_{t_2} = \frac{B_{t_2}}{\mu_2}$$

$$\text{But} \quad H_{t_1} = H_{t_2} \qquad \text{(boundary condition)}$$

$$\frac{B_{t_1}}{\mu_1} = \frac{B_{t_2}}{\mu_2}$$

$$\mu_2 B_{t_1} = \mu_1 B_{t_2}$$

From the Fig.3.19,

$$B_{t_1} = B_1 \sin\theta_1$$

$$B_{t_2} = B_2 \sin\theta_2$$

Substituting these values in the above equation,

$$\mu_2 B_1 \sin\theta_1 = \mu_1 B_2 \sin\theta_2 \qquad \ldots (2)$$

Dividing equation (2) by (1),

$$\frac{\mu_2 B_1 \sin\theta_1}{B_1 \cos\theta_1} = \frac{\mu_1 B_2 \sin\theta_2}{B_2 \cos\theta_2}$$

$$\mu_2 \tan\theta_1 = \mu_1 \tan\theta_2$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}$$

Similarly, if the flux ϕ_{21} is produced by second coil current i_2, the induced emf v_1 in coil 1 is proportional to the rate of change of current i_2.

$$v_1 \propto \frac{di_2}{dt}$$

$$v_1 = M\frac{di_2}{dt}$$

From Faraday's law,

$$v_1 = N_1\frac{d\phi_{21}}{dt}$$

Equating these two equations,

$$M = N_1\frac{d\phi_{21}}{di_2}$$

If the permeability is constant, then

$$M = \frac{N_1\ \phi_{21}}{i_2}$$

The mutual inductance between two coils is defined as the ratio of induced magnetic flux linkage in one coil to the current through in other coil.

3.15.3. Coefficient of Coupling

The fraction of the total flux produced by one coil linking a second coil is called the coefficient of coupling (K).

$$K = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

The mutual inductance is given by

$$M = \frac{N_2\ \phi_{12}}{i_1} \text{ and } M = \frac{N_1\ \phi_{21}}{i_2}$$

$$M^2 = \left(\frac{N_2\ \phi_{12}}{i_1}\right)\left(\frac{N_1\ \phi_{21}}{i_2}\right)$$

Substituting the value of ϕ_{12} and ϕ_{21} in terms of K

$$M^2 = \left(\frac{N_2\,K\,\phi_1}{i_1}\right)\left(\frac{N_1\,K\,\phi_2}{i_2}\right)$$

$$= K^2\left(\frac{N_2\,\phi_1}{i_1}\right)\left(\frac{N_1\,\phi_2}{i_2}\right)$$

$$M^2 = K^2 L_1 L_2 \qquad \left[\because L_1 = \left(\frac{N_1\,\phi_1}{i_1}\right) \& \ L_2 = \left(\frac{N_2\,\phi_2}{i_2}\right)\right.$$

$$\therefore\ M = K\sqrt{L_1 L_2}$$

$$\therefore\ K = \frac{M}{\sqrt{L_1 L_2}}$$

Since $\phi_{12} < \phi_1$ or $\phi_{21} < \phi_2$, the $M < \sqrt{L_1 L_2}$

i.e., K is always less than one. $K < 1$.

3.15.4. Coupled Coils in Series

Series Connection (Aiding)

Consider two coupled coils of self inductances L_1 and L_2. Let M be the mutual inductance between them. The two coils are connected in series (aiding) as shown in Fig.3.21.

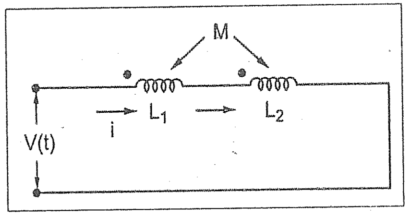


Fig. 3.21. Series circuits (aiding)

Since, the current is entering both the coils at the dotted terminal, it is called series aiding connection. *i.e.,* M has the same sign as L.

By applying Kirchhoff's law,

$$L_1\frac{di}{dt} + M\frac{di}{dt} + L_2\frac{di}{dt} + M\frac{di}{dt} = v(t)$$

$$(L_1 + L_2 + 2M)\frac{di}{dt} = v(t)$$

The equivalent inductance $L = L_1 + L_2 + 2M$

Series Connection (Opposing)

The two coils are connected in series (opposing) as shown in Fig.3.22. Since the current is entering first coil at dotted terminal and leaving other coil at dotted terminal, it is called series opposing connection *i.e.*, M has the opposite sign as L.

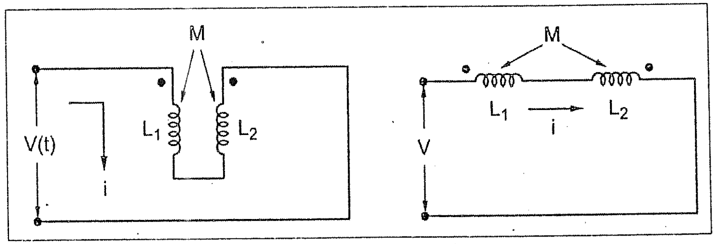


Fig. 3.22. Series circuit (opposing)

By applying Kirchhoff's law

$$L_1 \frac{di}{dt} - M\frac{di}{dt} + L_2\frac{di}{dt} - M\frac{di}{dt} = v(t)$$

$$(L_1 + L_2 - 2\,M)\frac{di}{dt} = v(t)$$

The equivalent inductance $L = L_1 + L_2 - 2\,M$

3.15.5. Coupled Coils in Parallel

Parallel Connection (Aiding)

Consider the two coils connected in parallel (aiding) as shown in Fig.3.23.

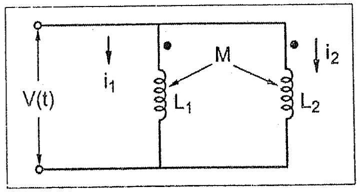


Fig. 3.23. Parallel circuit (aiding)

Let i_1 and i_2 be the currents flowing through coil 1 and coil 2.

Both the currents i_1 and i_2 enter the coils at the dotted terminals. By applying Kirchhoff's law,

$$L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} = v(t)$$

$$M\frac{di_1}{dt} + L_2\frac{di_2}{dt} = v(t)$$

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Taking the Laplace Transform for both the equations,

$$L_1\,s\,I_1(s) + M\,s\,I_2(s) = V(s)$$

$$M\,s\,I_1(s) + L_2\,s\,I_2(s) = V(s)$$

$$\begin{bmatrix} L_1 s & M s \\ M s & L_2 s \end{bmatrix}\begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} V(s) \\ V(s) \end{bmatrix}$$

Solving these equations, $$I_1(s) = \frac{\begin{vmatrix} V(s) & M s \\ V(s) & L_2 s \end{vmatrix}}{\begin{vmatrix} L_1 s & M s \\ M s & L_2 s \end{vmatrix}}$$

$$I_1(s) = \frac{V\,(L_2 - M)}{s\,(L_1 L_2 - M^2)}$$

$$I_2(s) = \frac{\begin{vmatrix} L_1 s & V(s) \\ M s & V(s) \end{vmatrix}}{\begin{vmatrix} L_1 s & M s \\ M s & L_2 s \end{vmatrix}}$$

$$I_1(s) = \frac{V\,(L_1 - M)}{s\,(L_1 L_2 - M^2)}$$

$$\text{Total current } I = I_1 + I_2$$

$$I = \frac{V\,(L_2 - M)}{s\,(L_1 L_2 - M^2)} + \frac{V\,(L_1 - M)}{s\,(L_1 L_2 - M^2)}$$

$$= \frac{V\,(L_2 + L_1 - 2\,M)}{s\,(L_1 L_2 - M^2)}$$

$$\frac{V}{I} = \frac{(L_1 L_2 - M^2)\, s}{L_1 + L_2 - 2\,M}$$

But $L s = \frac{V}{I}$

Equivalent inductance $L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2\,M}$

Parallel Connection (Opposing)

Consider the two coils connected in parallel (opposing) as shown in Fig.3.24.

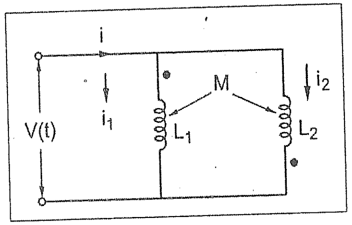


Fig. 3.24. Parallel circuit (opposing)

The current i_1 is entering the first coil L_1 at dotted terminal and the current i_2 is leaving the other coil at dotted terminal.

By applying Kirchhoff's law, $L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} = v(t)$

$$-M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = v(t)$$

By solving these two equations and the equivalent inductance is determined as

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2\,M}$$

3.15.6. Inductance of Solenoid

Consider a solenoid of N number of turns carrying the current I. If B is the flux density and A is the area of cross-section of the solenoid, then flux linkage through the solenoid is $N\phi = NBA$. $[\because \phi = BA]$.

$$L = \frac{N\phi}{I} = \frac{NBA}{I}$$

But for long solenoid $B = \frac{\mu_0 N I}{l}$

Substituting the value of B in the above equation

$$L = \frac{N A}{I} \cdot \left(\frac{\mu_0 N I}{l}\right) = \frac{\mu_0 N^2 A}{l}$$

3.15.7. Inductance of Toroid

Consider a toroid of N number of turns carrying current I with mean radius R as shown in Fig.3.25.

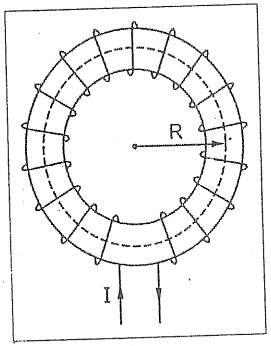


Fig. 3.25. Toroid

If B is the flux density in the toroid, then

$$B = \frac{\mu_0 N I}{l}$$

where, l is the mean length of the coil.

$$l = 2\pi R$$

$$B = \frac{\mu_0 N I}{2\pi R}$$

The flux linkage in the toroid is $N\phi$.

$$N\phi = N B A$$

$$= N A \frac{\mu_0 N I}{2\pi R} = \frac{\mu_0 N^2 A I}{2\pi R}$$

where, A is the area of cross-section of coil.

If r is the radius of the coil, $A = \pi r^2$.

$$N\phi = \frac{\mu_0 N^2 I \cdot \pi r^2}{2\pi R} = \frac{\mu_0 N^2 r^2 I}{2 R}$$

Inductance of the toroid is $L = \frac{N\phi}{I} = \frac{\mu_0 N^2 r^2}{2 R}$

$$L = \frac{\mu_0 N^2 r^2}{2 R}$$

3.15.8. Inductance of Coaxial Cable

Consider a coaxial cable of inner radius 'a' and outer radius 'b' as shown in Fig.3.26, Let I be the current in inner cylinder and – I be the current in outer cylinder.

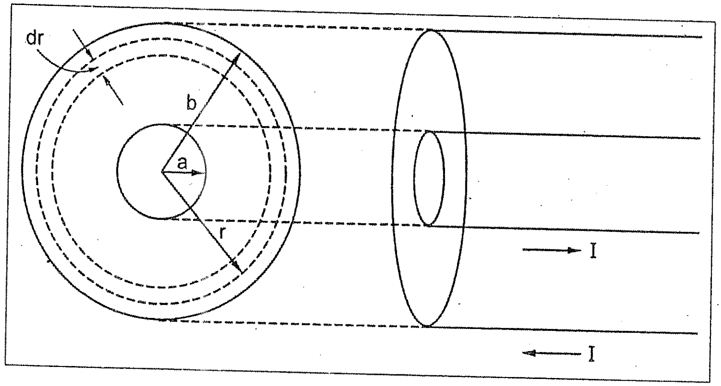


Fig. 3.26. Coaxial cable

Consider a annular ring of thickness dr at a distance r from the centre of the cable. The flux density B is given by

$$B = \frac{\mu_0 I}{2\pi r}$$

The total flux linkage per unit length between a and b,

$$\phi = \int_a^b \frac{\mu_0 I}{2\pi r} \cdot dr = \frac{\mu_0 I}{2\pi} [\, ln\, r\,]_a^b$$

$$= \frac{\mu_0 I}{2\pi} ln\left(\frac{b}{a}\right)$$

Inductance of coaxial cable per unit length is given by

$$L = \frac{\phi}{I}$$

$$L = \frac{\mu_0}{2\pi} ln\left(\frac{b}{a}\right)$$

3.15.9. Inductance of Coaxial Cable with Solid Inner Conductor

Consider a coaxial cable with solid inner conductor of radius 'a' and outer radius 'b'. Let I be the current in solid conductor and – I be the current in outer conductor.

The flux density within the solid conductor at a distance r from axis of the cable is

$$B = \frac{\mu_0 \mu_r I' r}{2\pi a^2}; \quad 0 < r < a$$

The current flowing in solid inner conductor between a to r is

$$I' = \frac{I}{\pi a^2} \cdot \pi r^2 = \frac{I}{a^2} r^2$$

$$\text{Now} \quad B = \frac{\mu_0 \mu_r r}{2\pi a^2} \cdot \frac{I r^2}{a^2} = \frac{\mu_0 \mu_r I r^3}{2\pi a^4}$$

The total flux linkage per unit length between 0 and a.

$$\phi = \int_0^a \frac{\mu_0 \mu_r I r^3}{2\pi a^4} \cdot dr = \frac{\mu_0 \mu_r I}{2\pi a^4} \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{\mu_0 \mu_r I}{2\pi a^4} \left[\frac{a^4}{4}\right]$$

$$\phi = \frac{\mu_0 \mu_r I}{8\pi}$$

Inductance of solid conductor between 0 and a.

$$L = \frac{\phi}{I}$$

$$L_1 = \frac{\mu_0 \mu_r}{8\pi}$$

Inductance of coaxial cable per unit length between a and b

$$L_2 = \frac{\mu_0}{2\pi} ln\left[\frac{b}{a}\right] \quad \text{(already derived)}$$

Inductance of coaxial cable per unit length is

$$L = L_1 + L_2$$

$$= \frac{\mu_0 \mu_r}{8\pi} + \frac{\mu_0}{2\pi} ln\left[\frac{b}{a}\right]$$

$$L = \frac{\mu_0}{4\pi}\left[\frac{1}{2}\mu_r + 2\,ln\left(\frac{b}{a}\right)\right] \text{ H/m}$$

3.15.10. Inductance of Two Transmission Line

Consider two conductors (wires) A and B of radius a and b respectively and separated by the distance d. The conductor A carries a current of I and conductor B carries a current of – I as shown in Fig.3.27.

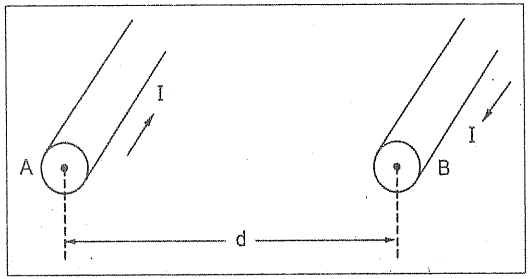


Fig. 3.27. Two Transmission line

The internal flux linkage of the conductor A is given by

$$\phi_1 = \frac{\mu_0 \mu_r I}{8\pi} \qquad \text{(refer previous)}$$

The external flux linkage with the conductor A is given by

$$\phi_2 = \frac{\mu_0 I}{2\pi} ln\left(\frac{d}{a}\right) \qquad \text{(refer previous)}$$

The total flux linkage of A is $\phi = \phi_1 + \phi_2$

$$= \frac{\mu_0 \mu_r I}{8\pi} + \frac{\mu_0 I}{2\pi} ln\left(\frac{d}{a}\right)$$

The total inductance of conductor A is

$$L_A = \frac{\phi}{I}$$

$$L_A = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2\,ln\left(\frac{d}{a}\right)\right] \text{ H/m}$$

Similarly for conductor B, the total flux linkage is

$$\phi = \frac{\mu_0 \mu_r I}{8\pi} + \frac{\mu_0 I}{2\pi} ln\left(\frac{d}{a}\right)$$

The total inductance of conductor B is

$$L_B = \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2\,ln\left(\frac{d}{a}\right)\right]$$

The loop inductance of transmission line per unit length is given by.

$$L = L_A + L_B$$

$$= \frac{\mu_0}{4\pi}\left[\frac{\mu_r}{2} + 2\,ln\left(\frac{d}{a}\right) + \frac{\mu_r}{2} + 2\,ln\left(\frac{d}{a}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\left[\mu_r + 2\,ln\left(\frac{d}{a}\right) + 2\,ln\left(\frac{d}{a}\right)\right]$$

$$L = \frac{\mu_0}{4\pi}\left[\mu_r + 2\,ln\left(\frac{d}{a}\right)^2\right] \text{ H/m}$$

$$\text{or} \quad L = \frac{\mu_0}{4\pi}\left[\mu_r + 4\,ln\,\frac{d}{a}\right]$$

3.16. ENERGY STORED IN MAGNETIC FIELD

When a current through an inductor is increased from 0 to I with the potential difference across the inductor is v, then the energy supplied by the source in time dt is given by

$$dW = v\,i\,dt$$

Energy stored in magnetic field is given by

$$W = \int_0^1 v\,i\,dt = \int_0^1 L\frac{di}{dt}\cdot i\,dt$$

$$= L\int_0^I i\,di = L\frac{I^2}{2}$$

$$W = \frac{1}{2}LI^2$$

Energy density

The energy stored in a magnetic field is given by

$$W = \frac{1}{2}LI^2$$

The inductance of the solenoid (for example) is given by

$$L = \frac{\mu_0 N^2 A}{l}$$

Substituting the value of L in the above equation

$$W = \frac{1}{2}\frac{\mu_0 N^2 A}{l} I^2$$

$$= \frac{1}{2}\mu_0\left(\frac{NI}{l}\right)^2 l\,A$$

$$W = \frac{1}{2}\mu_0 H^2 l\,A \qquad \left[\because H = \frac{NI}{l}\right]$$

Energy stored per unit volume

$$W = \frac{1}{2}\mu_0 H^2 \quad J/m^3 \qquad [\because \text{Volume} = l\,A]$$

$$= \frac{1}{2}(\mu_0 H)H$$

Magnetic energy density

$$W = \frac{1}{2}BH \quad \text{joules/m}^3$$

The energy stored in a magneto static field is

$$W = \int_v w\,dv$$

$$W = \frac{1}{2}\int_v B\cdot H\cdot dv = \frac{1}{2}\int \mu H^2\,dv$$

SOLVED EXAMPLES

Example 3.1 *An iron ring with a cross sectional area of 3 cm^2 and a mean circumference of 15 cm is wound with 250 turns wire carrying a current of 0.3 A. The relative permeability of the ring is 1500. Calculate the flux established in the ring.*

☺***Solution:***

Area, $A = 3\text{ cm}^2 = 3\times10^{-4}$ m

Mean circumference $2\pi r = 15\text{ cm} = 0.15$ m

$N = 250,\ \mu_r = 1500,\ A = 3\text{ cm}^2 = 3\times10^{-4}\text{ m}^2$

$I = 0.3$ A

$$\text{The flux established in the coil} = \frac{\mu_0\mu_r NAI}{2\pi r}$$

$$= \frac{4\pi\times10^{-7}\times1500\times250\times3\times10^{-4}\times0.3}{0.15}$$

$$= \mathbf{282.743\ \mu\ webers}$$ ☺☺

Example 3.2 *A round copper conductor is carrying a current of 250 A. Determine the magnetising force and flux density at a distance of 10 cm from the conductor.*

☺***Solution:*** $I = 250$ A, $r = 10\text{ cm} = 0.1$ m

$$\text{Flux density } B = \frac{\mu_0 NI}{2\pi r} \qquad [N = 1 \text{ turn}]$$

$$B = \frac{4\pi\times10^{-7}\times1\times250}{2\pi\times0.1}$$

$$B = 0.5\times10^{-3}\ \text{Wb/m}^2$$

$$\mathbf{B = 0.5\ mWb/m^2}$$

$$\text{Magnetic force } H = \frac{B}{\mu_0}$$

$$\mathbf{H} = \frac{\mathbf{0.5\times10^{-3}}}{\mathbf{4\pi\times10^{-7}}} = \mathbf{397.88\ AT/m}$$ ☺☺

Example 3.3 *Determine the magnetic flux density both inside and outside an infinitely long, straight conductor with a circular cross-section of radius 'a' carrying a steady current I. Plot the variation of the flux density with radial distance.*

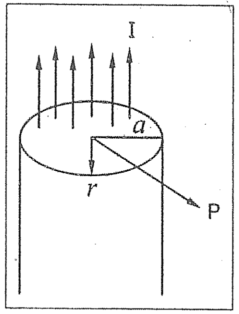
☺***Solution:*** Consider a solid cylindrical conductor of radius 'a', infinite in length, carries a steady state current of I.

The current density of the conductor in J

$$J = \frac{I}{\pi a^2} \text{ A/m}$$

Consider any point P at a radial distance r inside the conductor where the magnetic flux density is to be determined.

For inside the conductor ($r \le a$).



Apply Ampere's law,

$$\oint H \cdot dl = I = J \cdot A$$

$$H \cdot 2\pi r = \frac{I}{\pi a^2} \cdot \pi r^2$$

$$H = \frac{I r}{2\pi a^2}$$

$$\text{Magnetic flux density } B = \mu H$$

$$B = \frac{\mu I r}{2\pi a^2}$$

At the surface of the conductor, $r = a$

$$B = \frac{\mu I}{2\pi a}$$

For outside the conductor ($r > a$).

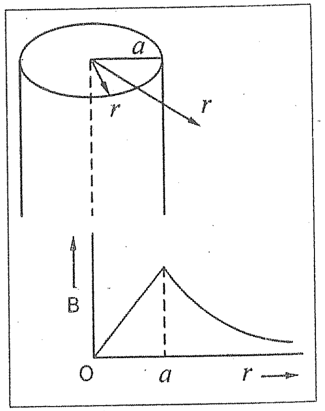
According to Ampere's law,

$$\oint H \cdot dl = I$$

$$H \cdot 2\pi r = I$$

$$H = \frac{I}{2\pi r}$$

$$B = \frac{\mu I}{2\pi r}$$



☺☺

Example 3.4 *A circular coil of radius 10 cm is made up of 100 turns. It carries a current of 5 A. Compute the magnetic field intensity at the centre of the coil.*

☺***Solution:***

$$a = 10 \times 10^{-2} \text{ m}, \quad N = 100, \quad I = 5 \text{ A}$$

$$H = \frac{N I}{2a} = \frac{100 \times 5}{2 \times 0.1} = \mathbf{2500 \ AT/m}$$

☺☺

Example 3.5 *Calculate the magnetic flux density due to circular coil of 100 ampere turns and area of 70 cm^2 on the axis of the coil at a distance of 10 cm from the centre.*

☺***Solution:***

$$N I = 100 \text{ AT}, \quad d = 0.10 \text{ m}$$

$$\pi a^2 = 70 \text{ cm}^2$$

$$\therefore \quad a^2 = 22.28 \text{ cm}^2 = 22.28 \times 10^{-4} \text{ m}$$

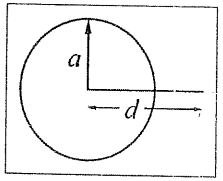
$$\text{Magnetic flux density } B = \frac{\mu_0 N I a^2}{2(a^2 + d^2)^{3/2}}$$

$$= \frac{4\pi \times 10^{-7} \times 100 \times 22.28 \times 10^{-4}}{2(22.28 \times 10^{-4} + 0.01)^{3/2}}$$

$$= 103.7 \times 10^{-6} \text{ Tesla} = \mathbf{103.7 \ \mu \ Tesla}$$

☺☺

Example 3.6 *A wire carrying a current of 100 A is bent into the form of a circle of diameter 10 cm. Calculate (a) flux density at the centre of the coil, (b) flux density at a point on the axis of the coil and 12 cm from it.*



☺***Solution:***

(a) Flux density at centre of the coil

$$B = \frac{\mu_0 N I}{2a}$$

Let $N = 1, \quad I = 100\ A$

Radius $a = \frac{10}{2} = 5\ cm$

$$= 5 \times 10^{-2}\ m$$

$$\therefore B = \frac{4\pi \times 10^{-2} \times 1 \times 100}{2 \times 5 \times 10^{-2}}$$

$$B = 1.256 \times 10^{-3}\ Wb/m^2 = \mathbf{1.256\ mWb/m^2}$$

(b) Flux density at a point on the axis of the coil and 12 cm from it

$$B = \frac{\mu_0 N I a^2}{2(a^2 + d^2)^{3/2}}$$

But $d = 12 \times 10^{-2}\ m$

Then $$B = \frac{4\pi \times 10^{-7} \times 100 \times (5 \times 10^{-2})^2}{2[(5 \times 10^{-2})^2 + (12 \times 10^{-2})^2]^{3/2}}$$

$$B = 0.0714 \times 10^{-3}\ Wb/m^2$$

$$\mathbf{B = 0.0714\ mWb/m^2}$$ ☺☺

Example 3.7 *Find H at the centre of an equivalent triangular loop of side 4 m carrying current of 5 A.*

☺***Solution:*** $I = 5\ A, \quad a = 4\ m$

$$h = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{3}{4}}\, a$$

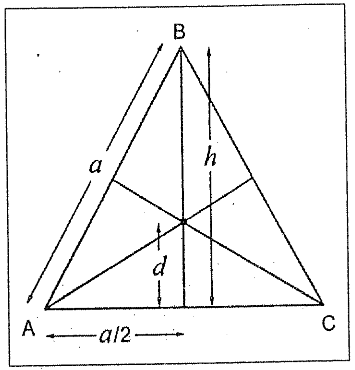
$$d = \frac{h}{3} = \sqrt{\frac{3}{4}} \cdot \frac{a}{3}$$

$$d = \frac{a}{\sqrt{12}}$$

Flux density B at centre due to AB = $\frac{\mu_0 I}{2\pi d}$

$$H = \frac{I}{2\pi d} = \frac{I\sqrt{12}}{2\pi \times a} = \frac{5\sqrt{12}}{2\pi \times 4}$$

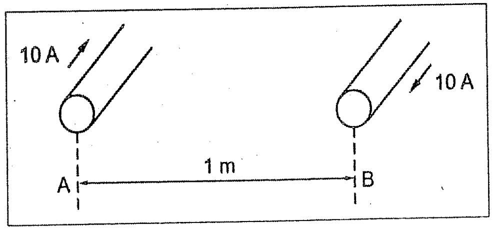
$$= 0.689\ A/m$$



Since it is a triangular loop carrying current, H due to AB, BC and CA.

$$\therefore H = 3 \times 0.689 = \mathbf{2.067\ A/m}$$ ☺☺

Example 3.8 *A single phase circuit comprises of two parallel conductors A and B have 1 cm diameter and spaced 1 metre apart. The conductors A and B carry currents of 10 A and − 10 A respectively. Determine the field intensity at the surface of each conductor and also in the middle of A and B.*



☺***Solution:*** Magnetic field intensity at the surface of B due to A and B

$$d_1 = 1\text{ m}, \quad d_2 = 0.5 \times 10^{-2}\text{ m}$$

$$H = \frac{I}{2\pi d_1} + \frac{I}{2\pi d_2}$$

$$H = \frac{I}{2\pi} + \frac{I}{2\pi \times 0.5 \times 10^{-2}}$$

$$= \frac{10}{2\pi}\left[1 + \frac{10^2}{0.5}\right]$$

$$= \frac{10^3}{\pi} = 318.3 \text{ A/m}$$

This is same as the magnetic field intensity at the surface of A due to A and B.

Magnetic field intensity at the mid-point of A and B.

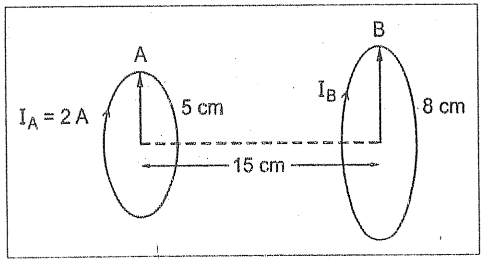
$$H = \frac{I}{2\pi \times 0.5} + \frac{I}{2\pi \times 0.5}$$

$$= \frac{2I}{2\pi \cdot (0.5)}$$

$$= \frac{2 \times 10}{\pi} = \mathbf{6.366 \text{ A/m}}$$

☺☺

Example 3.9 *Two narrow circular coils A and B have a common axis and are placed 15 cm apart. Coil A has 10 turns of radius 5 cm with a current of 2 A passing through it. Coil B has a single turn of radius 8 cm. If the magnetic field at the centre of coil A is to be zero, what current must be passed through coil B.*



☺***Solution:***

For coil A	For coil B
$N_A = 10$	$N_B = 1$
$a = 5 \times 10^{-2}$ m	$b = 8 \times 10^{-2}$ m
$I_A = 2$ A	$I_B = ?$

$$d = 15 \times 10^{-2}\text{ m}$$

The magnetic field at the centre of coil A is due to coil A is

$$H_A = \frac{NI}{2a} = \frac{10 \times 2}{2 \times 5 \times 10^{-2}}$$

$$= 200 \text{ AT/m}$$

But the field H_A is zero. This is possible only when the magnetic field H_B due to the coil B at the centre of coil A opposes H_A is equal and opposite direction.

$$\text{i.e.,} \quad |H_A| = |H_B|$$

The magnetic field at the centre of coil A due to coil B is

$$H_B = \frac{N_B b^2 I_B}{2(b^2 + d^2)^{3/2}}$$

$$= \frac{(8 \times 10^{-2})^2 I_B}{2[(8 \times 10^{-2})^2 + (15 \times 10^{-2})^2]^{3/2}}$$

$$= \frac{64 \times 10^{-4} I_B}{9.826 \times 10^{-3}} = 0.651 I_B$$

$$\text{But} \quad |H_A| = |H_B|$$

$$200 = 0.651 I_B$$

$$\mathbf{I_B = 307.06 \text{ A}}$$

☺☺

Example 3.10 *Two circular coils are located at z = 0 plane and z = 5 m plane centered about the axis. The first coil having a radius of 1 m carries a current of 10 A. The second coil having a radius of 0.5 m carries a current of 20 A. Calculate the magnetic field intensity at (0, 0, 2.5 m).*

☺**Solution:** $I_1 = 10$ A, $a = 1$ m, $h = 2.5$ m,

H_1 due to coil in $z = 0$ plane is

$$H_1 = \frac{I_1\, a^2}{2\,(a^2+h^2)^{3/2}}\,\bar{a}_z = \frac{10}{2\,(1^2+2.5^2)^{3/2}}\,\bar{a}_z$$

$$H_1 = 0.25613\;\bar{a}_z \quad \text{A/m}$$

H_2 due to coil in $z = 5$ plane is

$$a = 0.5 \text{ m}$$

$$h = 2.5 \text{ m}$$

$$I_2 = 20 \text{ A}$$

$$H_2 = \frac{I\, a^2}{2\,(a^2+h^2)^{3/2}}\,(-\bar{a}_z)$$

$$H_2 = -0.1508\;\bar{a}_z \quad \text{A/m}$$

$$\therefore \text{ Resultant H} = H_1 + H_2 = 0.25613 - 0.1508$$

$$= 0.10533\;\bar{a}_z \quad \text{A/m}$$

If I_2 opposes I_1, resultant $H = H_1 - H_2 = \mathbf{0.407\,\bar{a}_z}$ **A/m.** ☺☺

Example 3.11 *Two current filaments are parallel to the z-axis one passes through (0, –0.5, 0) and carries 10 A in $\bar{a}_z$ direction and the other passes through (0, 0.5, 0) and carries 100 A in $\bar{a}_z$ direction. Find the H_x on the y-axis.*

☺**Solution:** H_1 due to the filament at (0, – 0.5, 0) is

$$H_1 = \frac{I_1}{2\pi d}$$

$$I_1 = 10 \text{ A}, \quad d = 1 \text{ m}$$

$$\therefore\; H_1 = \frac{10}{2\pi \times 1} = 1.59 \quad \text{A/m}$$

H_2 due to the filament at (0, 0.5, 0) is

$$H_2 = \frac{I_2}{2\pi d}$$

$$I_2 = 100 \text{ A}, \quad d = 1 \text{ m}$$

$$\therefore\; H_2 = \frac{100}{2\pi \times 1}$$

$$= 15.9 \quad \text{A/m}$$

$$\therefore \text{ Net field is } H_x = H_2 - H_1$$

$$\doteq 14.3307 \quad \text{A/m}$$

$$\therefore\; \mathbf{H_x = 14.3307 \quad A/m}$$ ☺☺

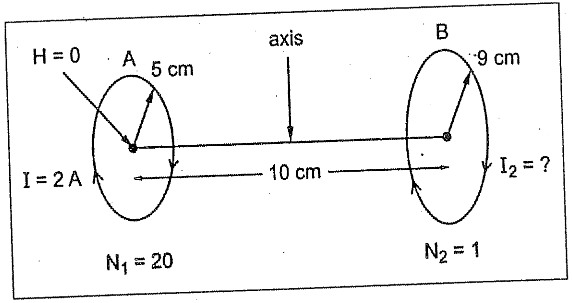
Example 3.12 *Two narrow circular coils A and B have a common axis and are placed 10 cm apart. Coil A has 20 turns of radius 5 cm, through which a current of 2 amperes flows. Coil B has a single turn of radius 9 cm. If the magnetic field at the centre of coil A is to be zero, what is the current in coil B?*

☺**Solution:**

$d = 10$ cm $= 0.1$ m

$N_1 = 20$ turns, $r_1 = 5 \times 10^{-2}$ m, $I_1 = 2$ A

$N_2 = 1$ turn, $r_2 = 9 \times 10^{-2}$ m, $I_2 = ?$



Magnetic field at the centre of coil A due to 2A current flowing in coil A is given by

$$H_1 = \frac{N_1\, I_1}{2\, r_1} = \frac{20 \times 2}{2 \times 5 \times 10^{-2}} = 400 \text{ A} \quad \text{turns/m}$$

Magnetic field at the centre of coil A due to current I_2 in coil B is given by

$$H_2 = \frac{N_2 \cdot I_2 \cdot r_2^2}{2(r_2^2 + d^2)^{3/2}} \text{ AT/m}$$

$$H_2 = \frac{(1) \cdot I_2 \cdot (81 \times 10^{-4})}{2(81 \times 10^{-4} + 0.01)^{3/2}}$$

$$= \frac{81 \times 10^{-4}\ I_2}{0.00487} = 1.66317\ I_2$$

For the field at centre of coil A to be zero, H_1 and H_2 should be equal and opposite.

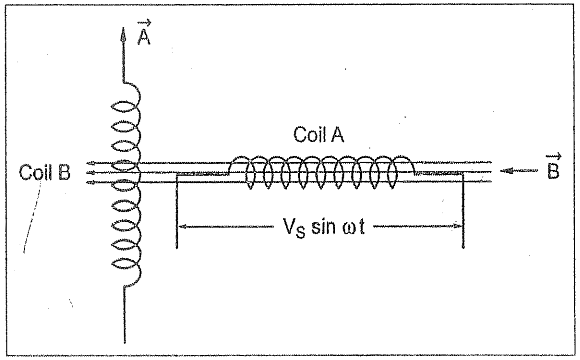
$$i.e., \quad |H_1| = |H_2|$$

$$1.66317\ I_2 = 400 \text{ AT/m}$$

$$I_2 = \frac{400}{1.66317} = 240.5 \text{ A}$$

∴ To make the field at centre of coil A to be zero, current in coil B should be 240.5 A. ☺☺

Example 3.13 *Two coils of wire, coil A and coil B, are placed with their axes perpendicular to each other in vicinity. Coil A is now excited by a time varying voltage from a source of EMF. If a flux has been successfully established around the coil A, what is the flux induced into the coil B.*



☺***Solution:*** Coil A and coil B are placed at an angle of 90° to each other. Coil A is excited by an alternating EMF source.

$$e = N\,[BA]\ \omega \cos(\omega t)$$

at their relative position B and A are at 90° to each other.

$$\phi = B \cdot A = B \cdot A \cos\theta = 0$$

So, as flux linked from coil A to coil B is zero. Hence EMF induced is zero. ☺☺

Example 3.14 *Two wires carrying in the same direction of 500 A and 800 A are placed with their axes 5 cm apart. Calculate the force between them.*

☺***Solution:***

$$I_1 = 500 \text{ A}, \quad I_2 = 800 \text{ A}, \quad d = 5 \times 10^{-2} \text{ m}$$

$$F = \frac{\mu_0 I_1 I_2}{4\pi d} = \frac{4\pi \times 10^{-7} \times 500 \times 800}{4\pi \times 5 \times 10^{-2}}$$

$$= \mathbf{0.8 \text{ Newtons}}$$ ☺☺

Example 3.15 *A solenoidal of length 30 cm is wound on a wooden core. The external and internal diameters of which are 4 cm and 3.5 cm respectively. If the current in the coil is 0.12 A, find the number of turns required to produce a magnetising force of 750 AT/m at the centre of the coil.*

☺***Solution:***

$$l = 0.3 \text{ m}$$

$$a = \frac{4 + 3.5}{2} = 3.75 \times 10^{-2} \text{ m (average radius)}$$

$$I = 0.12 \text{ A}, \quad H = 750 \text{ AT/m}$$

$$H = \frac{NI}{l}\left[\frac{\frac{l}{2}}{\sqrt{(l/2)^2 + a^2}}\right]$$

$$750 = \frac{N \times 0.12}{0.3}\,[0.97]$$

$$N = \frac{750 \times 0.3}{0.97 \times 0.12} = \mathbf{1933 \text{ turns}}$$ ☺☺

Example 3.16 *The magnetic field strength of a 200 turn coil excited by a current of 2A is to be determined. The length of the solenoid is 0.25 m. The direction of the field strength is already known.*

☺***Solution:*** Solenoid length, $l = 0.25$ m

Number of turns, $N = 200$

Current in solenoid, $I = 2$ A

Now, the field strength in a solenoid,

$$|H| = \frac{NI}{l} = \frac{200 \times 2}{0.25} = \mathbf{1600\ AT/m}$$ ☺☺

Example 3.17 *Determine the force between two parallel conductors of length 1 m separated by 50 cm in air and carrying currents of 30 A. (a) in the same direction, (b) in the opposite direction.*

☺***Solution:*** $I = 30$ A, $l = 1$ m, $d = 0.5$ m

(a) Force of attraction

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi d} = \frac{\mu_0 I^2 l}{2\pi d}$$

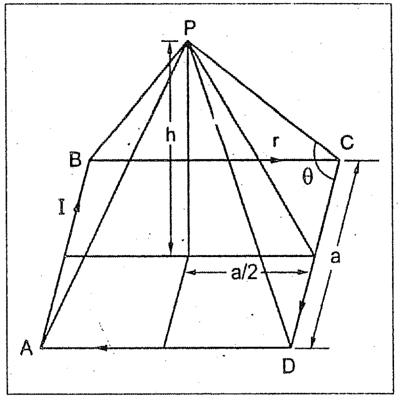
$$= \frac{4\pi \times 10^{-7} \times (30)^2 \times 1}{2\pi \times 0.5} = \mathbf{0.36 \times 10^{-3}\ Newtons}$$

(b) Force of repulsion

$$F = \frac{\mu_0 I^2 l}{2\pi d} = \mathbf{0.36 \times 10^{-3}\ Newtons}$$ ☺☺

Example 3.18 *A steady direct current I amps flows in a wire bent in the form of a square of side a. Assuming that the Z-axis passing through the centre of the square is normal to the plane of the square, find the magnetic field intensity H at any point on the axis.*

☺***Solution:*** Consider a rectangular coil of side '*a*' carrying current I. Let P be a point on the axis of the rectangular coil at a distance *h* from the centre of the coil as shown in Fig.



Magnetic flux density at point P on the axis of a rectangular coil due to side AB.

$$B_1 = \frac{\mu_0 I}{4\pi r}(\cos\theta_1 + \cos\theta_2) \quad \text{[for finite current carrying conductor]}$$

$$\text{where } r = \sqrt{\frac{a^2}{2} + h^2}$$

$$\theta_1 = \theta_2 = \theta$$

$$B_1 = \frac{\mu_0 I}{4\pi\sqrt{\frac{a^2}{2} + h^2}} 2\cos\theta$$

$$\text{where } \cos\theta = \frac{a/2}{\sqrt{(a/2)^2 + h^2}}$$

$$B_1 = \frac{\mu_0 I}{2\pi\sqrt{\frac{a^2}{2} + h^2}} \frac{a/2}{\sqrt{(a/2)^2 + h^2}}$$

$$= \frac{\mu_0 I a}{4\pi\sqrt{\frac{a^2}{2} + h^2}\sqrt{(a/2)^2 + h^2}}$$

Magnetic flux density at P due to both sides AB and CD

$$B' = 2B_1 = \frac{\mu_0 I a}{2\pi\sqrt{\frac{a^2}{2} + h^2}\sqrt{(a/2)^2 + h^2}}$$

Similarly the magnetic flux density at P due to AD and BC

$$B'' = 2\,B_2 = \frac{\mu_0 I a}{2\pi\sqrt{\frac{a^2}{2} + h^2}\ \sqrt{(a/2)^2 + h^2}}$$

Total magnetic field due to (all sides) whole rectangular coil

$$B = B' + B''$$

$$B = \frac{\mu_0 I a}{\pi\sqrt{\frac{a^2}{2} + h^2}\ \sqrt{(a/2)^2 + h^2}}$$

Total magnetic field due to rectangular coil at a distance h from centre of the coil.

$$H = \frac{I a}{\pi\sqrt{\frac{a^2}{2} + h^2}\ \sqrt{(a/2)^2 + h^2}}$$

If $h = 0$ *i.e.,* the point P is at the centre of the coil, then

$$B = \frac{\mu_0 I a}{\pi \frac{a}{\sqrt{2}} \cdot \frac{a}{2}} = \frac{2\sqrt{2}\ \mu_0 I}{\pi a}$$

$$B = \frac{2\sqrt{2}\ \mu_0 I}{\pi a}\ \text{W/m}^2$$

$$H = \frac{2\sqrt{2}\ I}{\pi a}\ \textbf{A/m}$$

☺☺

Example 3.19 *What is the maximum torque on a square loop of 1000 turns in a field of flux density of 1 tesla. The loop has 10 cm sides and carries 3 A. What is the magnetic moment of the loop?*

☺Solution:

$N = 1000, \quad a = 0.1\text{ m}, \quad I = 3\text{ A}$

$B = 1\text{ T}, \quad A = a^2 = (0.1)^2$

$\text{Torque} = I A B$

$= 3 \times (0.1)^2 \times 1 = 0.03$ Newton-metre

Magnetic moment, $m = I A$

$= 3 \times 0.01 = \mathbf{0.03\ A.m^2}$

☺☺

Example 3.20 *A single twin of wire in the form of a square of side 10 cm carries a current of 20 A. Calculate the magnetic flux density produced by this circuit at a point on the central normal to its plane distance 20 cm from the plane.*

☺Solution: $I = 20\text{ A}, \quad a = 0.1\text{ m}, \quad h = 0.2\text{ m}$

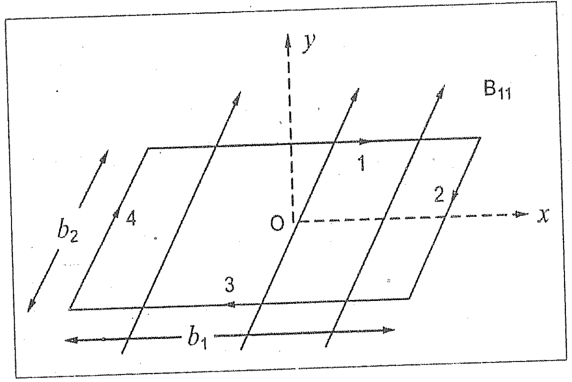
$$B = \frac{I a}{\pi\sqrt{\frac{a^2}{2} + h^2}\ \sqrt{\left(\frac{a}{2}\right)^2 + h^2}}$$

$$= \frac{20 \times 0.1}{\pi\sqrt{\frac{(0.1)^2}{2} + (0.2)^2}\ \sqrt{\left(\frac{0.1}{2}\right)^2 + (0.2)^2}}$$

$= \mathbf{1.522\ Tesla}$

☺☺

Example 3.21 *A rectangular loop in the xy plane with sides b_1 and b_2 carrying a current I lies in a uniform magnetic field $B = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$. Determine the force and torque on the loop.*



☺Solution: Consider a rectangular loop of sides b_1 and b_2 carrying a current I in a uniform magnetic field

$$B = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z$$

$$B_1 = B_z\,\bar{a}_z$$

$$B_{11} = B_x\,\bar{a}_x + B_y\,\bar{a}_y$$

Force acting on each side of the loop

$$\text{Force on side 1} = I\, b_1\, \bar{a}_x \times (B_x\, \bar{a}_x + B_y\, \bar{a}_y)$$

$$F_1 = \bar{a}_z\, I\, b_1\, B_y$$

$$\text{Force on side 2} = I\, b_2\, (-\bar{a}_y) \times (B_x\, \bar{a}_x + B_y\, \bar{a}_y)$$

$$F_2 = \bar{a}_z\, I\, b_2\, B_x$$

$$\text{Force on side 3} = I\, b_1\, (-\bar{a}_x) \times (B_x\, \bar{a}_x + B_y\, \bar{a}_y)$$

$$F_3 = -\bar{a}_z\, I\, b_1\, B_y = -F_1$$

$$\text{Force on side 4} = I\, b_2\, (\bar{a}_y) \times (B_x\, \bar{a}_x + B_y\, \bar{a}_y)$$

$$F_4 = -\bar{a}_z\, I\, b_2\, B_x$$

$$= -F_2$$

The net force acting on the loop is

$$F = F_1 + F_2 + F_2 + F_3 = \mathbf{0}$$

Torque on sides 1 and 3, $T_{13} = \bar{a}_x\, I\, b_1\, b_2\, B_y$

Torque on sides 2 and 4, $T_{24} = -\bar{a}_y\, I\, b_1\, b_2\, B_x$

Total torque on the rectangular loop is

$$T = T_{13} + T_{24}$$

$$= \mathbf{I\, b_1 b_2\, (\bar{a}_x\, B_y - \bar{a}_y\, B_x)}$$

☺☺

Example 3.22 *At a point P(x, y, z) the components of vector magnetic potential $\bar{A}$ are given as $A_x = 4x + 3y + 2z$, $A_y = 5x + 6y + 3z$ and $A_z = 2x + 3y + 5z$. Determine B at the point P.*

☺Solution:

$$B = \nabla \times A = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4x+3y+2z & 5x+6y+3z & 2x+3y+5z \end{bmatrix}$$

$$= \bar{a}_x\left[\frac{\partial}{\partial y}(2x+3y+5z) - \frac{\partial}{\partial z}(5x+6y+3z)\right] +$$

$$\bar{a}_y\left[\frac{\partial}{\partial z}(4x+3y+2z) - \frac{\partial}{\partial x}(2x+3y+5z)\right] +$$

$$\bar{a}_z\left[\frac{\partial}{\partial x}(5x+6y+3z) - \frac{\partial}{\partial y}(4x+3y+2z)\right]$$

$$= \bar{a}_x[3-3] + \bar{a}_y[2-2] + \bar{a}_z[5-3]$$

$$\mathbf{B = 2\,\bar{a}_z}$$

☺☺

Example 3.23 *In cylindrical co-ordinates $A = 50\,\rho^2\, \vec{a}_z$ Wb/m is a vector magnetic potential in a certain region of free space. Find the magnetic field intensity (H), magnetic flux density (B) and current density (J).*

☺Solution: $A = 50\,\rho^2\, \vec{a}_z$ Wb/m

$$B = \nabla \times A = \frac{1}{\rho}\begin{vmatrix} \vec{a}_\rho & \rho\,\vec{a}_\phi & \vec{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & 0 & 50\,\rho^2 \end{vmatrix}$$

$$= \frac{1}{\rho}\,\vec{a}_\rho\left[\frac{\partial}{\partial \phi}(50\,\rho^2)\right] - \vec{a}_\phi\left[\frac{\partial}{\partial \rho}(50\,\rho^2)\right]$$

$$= 0 - 100\,\rho\,\vec{a}_\phi$$

$$\mathbf{B = -100\,\rho\,\vec{a}_\phi\ Wb/m^2}$$

$$H = \frac{B}{\mu_0} = \frac{-100}{\mu_0}\,\rho\,\vec{a}_\phi = \mathbf{\frac{-100}{4\pi \times 10^{-7}}\,\rho\,\vec{a}_\phi\ A/m}$$

$$J = \nabla \times H$$

$$= \frac{1}{\rho}\begin{vmatrix} \vec{a}_\rho & \rho\,\vec{a}_\phi & \vec{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\rho\,(-100\,\rho)}{\mu_0} & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\vec{a}_\rho\left[0 - \frac{\partial}{\partial z}\frac{(-100\,\rho^2)}{\mu_0}\right] - 0 + \frac{1}{\rho}\vec{a}_z\left[\frac{\partial}{\partial \rho}\frac{(-100\,\rho^2)}{\mu_0}\right]$$

$$= 0 - \frac{200}{\mu_0}\vec{a}_z$$

$$\mathbf{J} = \frac{-200}{\mu_0}\vec{a}_z \text{ A/m}^2$$

☺☺

Example 3.24 *Why is it so that the magnetic vector potential is often referred to and electric vector potential is virtually unheard of in the field theory? How is the magnetic vector potential defined?*

If the magnetic vector potential of a magnetic field due to a solenoid is determined to be, at any point (x, y, z) in field,

$$5\,\bar{i}\,/\,(x^2 + y^2 + z^2)$$

then, What is the magnetic flux density in Weber/m² at any point in the rectangular co-ordinate system.

☺***Solution:*** The source of magnetic field unlike of electric field have direction associated with them.

The current elements have direction as well as magnitudes and it is necessary that additional information on the direction of the source be contained in the potential due to source. If 'magnetic vector potential' is designated by a vector $\bar{A}$, then it should be possible to contain $\bar{B}$ and $\bar{H}$ as the space derivative of $\bar{A}$. There are two possible space-derivatives operation on a vector quantity, namely 'divergence' and the 'curl'. The divergence operation yields a scalar quantity whereas the curl operation leads to a vector quantity.

$$\therefore\ \bar{B} = \nabla \times \bar{A}$$

Given:

$$\bar{A} = \frac{5\,\bar{i}}{(x^2+y^2+z^2)}$$

From the derivative of curl

$$\bar{B} = \nabla \times \bar{A} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{5}{x^2+y^2+z^2} & 0 & 0 \end{vmatrix}$$

$$\bar{B} = \bar{a}_y\left[\frac{\partial}{\partial z}\frac{5}{(x^2+y^2+z^2)}\right] + \bar{a}_z - \left[\frac{\partial}{\partial y}\frac{5}{(x^2+y^2+z^2)}\right]$$

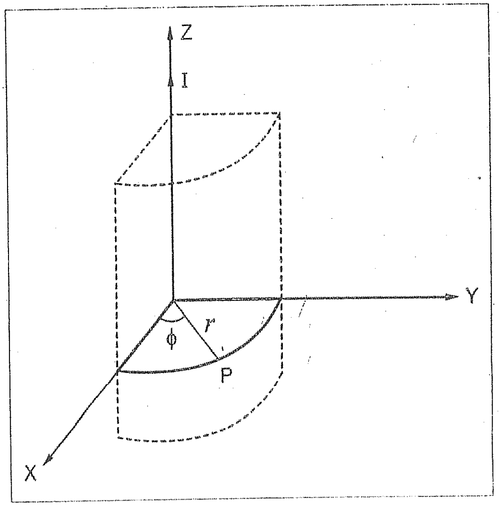
$$\bar{B} = 5\left[\frac{-2z\,\bar{a}_y}{(x^2+y^2+z^2)^2} - \frac{\bar{a}_z\,(-2y)}{(x^2+y^2+z^2)^2}\right]$$

$$\bar{B} = \frac{-10\,z}{(x^2+y^2+z^2)^2}\bar{a}_y + \frac{10\,y}{(x^2+y^2+z^2)^2}\bar{a}_z$$

☺☺

Example 3.25 *A very long, straight conductor located along the z-axis carries a current I in the z-direction. Obtain an expression for the magnetic vector potential at a point in the bisecting plane of the conductor. What is the magnetic flux density at that point?*

☺***Solution:***



Consider a very long, straight conductor located along the z-axis carries a current I in the z direction. Let P be any point (x, y) in the bisecting plane of the conductor at a distance r from the origin.

The magnetic flux density at a point P is given by

$$B = \frac{\mu_0 I}{2\pi r} \bar{a}_\phi$$

where μ_0 is the permeability of free space

$$r = \sqrt{x^2 + y^2}$$

The vector magnetic potential is given by

$$\nabla \times A = B$$

where, A is the vector magnetic potential.

$$\nabla \times A = B = \frac{\mu_0 I}{2\pi r} \bar{a}_\phi$$

Since B possesses only $\bar{a}_\phi$ component, only the ϕ component of the cylindrical curl is equated.

$$\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

Since A is independent of z,

$$\frac{\partial A_r}{\partial z} = 0$$

$$-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

$$\partial A_z = -\frac{\mu_0 I}{2\pi r} \cdot \partial r$$

Integrating,

$$\int \partial A_z = -\frac{\mu_0 I}{2\pi} \int \frac{1}{r}\, dr$$

$$A_z = -\frac{\mu_0 I}{2\pi}\, ln\ r + C$$

The constant of integration permits the location of zero reference with $A_z = 0$ at $r = r_0$. (Boundary condition).

$$0 = -\frac{\mu_0 I}{2\pi}\, ln\ r_0 + C$$

$$C = \frac{\mu_0 I}{2\pi}\, ln\ r_0$$

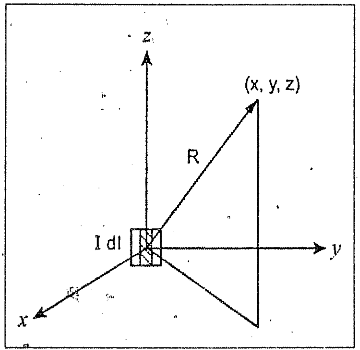
Substituting this constant value in A_z equation,

$$A_z = \frac{\mu_0 I}{2\pi}\, ln\ r + \frac{\mu_0 I}{2\pi}\, ln\ r_0$$

$$A_z = \frac{\mu_0 I}{2\pi}\, ln\left(\frac{r_0}{r}\right) \bar{a}_z$$

☺☺

Example 3.26 *Using the concept of magnetic vector potential, derive Biot-savart's law and Ampere's law.*



☺***Solution:*** The differential vector potential is

$$dA = \frac{\mu_0 I\, dl}{4\pi R} \bar{a}_z = \frac{\mu_0 I\, dl \cdot \bar{a}_z}{4\pi (x^2 + y^2 + z^2)^{1/2}}$$

Differential magnetic flux density

$$dB = \nabla \times dA$$

$$= \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 I\, dl}{4\pi \sqrt{(x^2+y^2+z^2)}} \end{bmatrix}$$

$$= \frac{\mu_0 I\, dl}{4\pi} \left[\frac{-y\,\bar{a}_x}{(x^2+y^2+z^2)^{3/2}} + \frac{x\,\bar{a}_y}{(x^2+y^2+z^2)^{3/2}} \right]$$

$$dB = \frac{\mu_0 I\, dl}{4\pi (x^2+y^2+z^2)^{3/2}} (-y\,\bar{a}_x + x\,\bar{a}_y)$$

This is Biot-savart law.

Differential magnetic field intensity,

$$dH = \frac{1}{\mu_0} dB = \frac{I\, dl}{4\pi (x^2+y^2+z^2)^{3/2}} (-y\,\bar{a}_x + x\,\bar{a}_y)$$

$$H = \int dH = \int \frac{I\, dl}{4\pi (x^2+y^2+z^2)^{3/2}} \cdot (-y\,\bar{a}_x + x\,\bar{a}_y)$$

$$\oint H \cdot dl = I$$

This is Ampere's law. ☺☺

Example 3.27 *Two coils A and B with 800 and 1200 turns respectively are having common magnetic circuit. A current of 0.5 A in coil A produces a flux of 3 mWb and 80% of flux links with coil B. Calculate L_1, L_2 and M.*

☺ ***Solution:***

Coil A

$N_1 = 800$

$I_1 = 0.5$ A

$\phi_1 = 3 \times 10^{-3}$ Wb

$K = 80\% = 0.8$

Coil B

$N_2 = 1200$

$\phi_{21} = 80\%$ of ϕ_1

$= 0.8 \times 3 \times 10^{-3}$ Wb

$$L_1 = \frac{N_1 \phi_1}{I_1} = \frac{800 \times 3 \times 10^{-3}}{0.5} = \mathbf{4.8\ Henry}$$ **Ans.** ☜

$$M = \frac{N_2 \phi_{21}}{I_1}$$

$$= \frac{1200 \times 0.8 \times 3 \times 10^{-3}}{0.5} = \mathbf{5.76\ Henry}$$ **Ans.** ☜

$$M = K\sqrt{L_1 L_2}$$

$$L^2 = \frac{M^2}{K^2 L_1} = \frac{(5.76)^2}{(0.8)^2 \times 4.8} = \mathbf{10.8\ Henry}$$ **Ans.** ☜ ☺☺

Example 3.28 *A solenoid consisting of 1000 turns of wire wound on a former of length 100 cm and diameter 3 cm is placed coaxially within another solenoid of the same length and number of turns but with a diameter of 6 cm. Find the mutual inductance and the coupling coefficient of the arrangement.*

☺ ***Solution:***

Solenoid 1

$N_1 = 1000$

$l_1 = 1$ m

$d_1 = 3 \times 10^{-2}$ m

Area $A_1 = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 3 \times 3 \times 10^{-4}$

$= 7.07 \times 10^{-4}$ m^2

$$L_1 = \frac{\mu_0 N_1^2 A_1}{l_1}$$

Solenoid 2

$N_2 = 1000$

$l_2 = 1$ m

$d_2 = 6 \times 10^{-2}$ m

$A_2 = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 6 \times 6 \times 10^{-4}$

$= 28.27 \times 10^{-4}$ m^2

$$L_2 = \frac{\mu_0 N_2^2 A_2}{l_2}$$

$$L_1 = \frac{4\pi \times 10^{-7} \times (1000)^2 \times 7.07 \times 10^{-4}}{1} = 888.4 \times 10^{-6}\ \text{Henry}$$

$$L_2 = \frac{4\pi \times 10^{-7} \times (1000)^2 \times 28.27 \times 10^{-4}}{1} = 3.55 \times 10^{-3}\ \text{Henry}$$

[Alternatively

$$L_2 = \left(\frac{N_2}{N_1}\right)^2 \left(\frac{A_2}{A_1}\right)^2 ;\ L_1 = \left(\frac{1000}{1000}\right)^2 \left(\frac{28.27 \times 10^{-4}}{7.07 \times 10^{-4}}\right) L_1 = 4\,L_1\ ]$$

Since the inner solenoid has half of the diameter as that of outer solenoid, the only 50% of flux produced by outer coil links with the inner coil.

Coupling coefficient $K = \mathbf{0.5}$ **Ans.** ☜

$$M = K\sqrt{L_1 L_2}$$

$$= 0.5\sqrt{888.4 \times 4 \times 10^{-6} \times 3.55 \times 10^{-3}}$$

$$= \mathbf{0.888 \times 10^{-3}\ Wb}$$ **Ans.** ☜ ☺☺

Example 3.29 *A and B are two coils with 1000 and 1500 turns respectively and lying in parallel planes, and it is found that 50% of the flux produced by coil A links with the coil B. The hysteresis curve indicates that a current of 3A in the coil A produces 0.3 m wb, while the same current in a coil B produces 0.6 m wb. Calculate the mutual inductance between the coils. Also calculate self inductances of the coils.*

☺ ***Solution:***

Coil A

$$N_A = 1000$$
$$\phi_A = 0.3 \times 10^{-3} \text{ Wb}$$
$$I_A = 3 \text{ A}$$
$$L_A = \frac{N_A \phi_A}{I_A} = \frac{1000 \times 0.3 \times 10^{-3}}{3}$$
$$L_A = 0.1 \text{ H}$$

Coil B

$$N_B = 1500$$
$$\phi_B = 0.6 \times 10^{-3} \text{ Wb}$$
$$I_B = 3 \text{ A}$$
$$L_B = \frac{N_B \phi_B}{I_B} = \frac{1500 \times 0.6 \times 10^{-3}}{3}$$
$$L_B = 0.3 \text{ H}$$

Since the flux linkage is 50%. K = 0.5.

$$\text{Mutual inductance } M = K\sqrt{L_1 L_2} = 0.5\sqrt{0.1 \times 0.3}$$
$$= \mathbf{0.0866 \text{ Henry}} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 3.30 *A circuit has 1000 turns enclosing a magnetic circuit 30 cm² in cross-section. A current of 5 A in the circuit produce a field of flux density 1 Tesla and when the current is doubled the flux density increases only by 50%. Determine the mean value of inductance of the circuit between 5 A and 10 A and also the induced emf when the current increases uniformly from 5 A to 10 A in 0.1 econd.*

☺ ***Solution:***

$$N = 1000$$
$$A = 30 \text{ cm}^2 = 30 \times 10^{-4} \text{ m}$$
$$B = 1 \text{ Tesla} = 1 \text{ Wb/m}^2$$
$$\phi = BA$$
$$= 1 \times 30 \times 10^{-4} = 30 \times 10^{-4} \text{ Wb}$$

$$\text{Flux } \phi_1 = 30 \times 10^{-4} \text{ Wb at } i_1 = 5 \text{ A}$$
$$\phi_2 = 1.5 \text{ times of } \phi_1 \text{ (50\% high) at } i_2 = 10 \text{ A}$$
$$= 1.5 \times 30 \times 10^{-4} \text{ Wb}$$
$$= 45 \times 10^{-4} \text{ Wb at } i_2 = 10 \text{ A}$$
$$\frac{d\phi}{di} = \frac{\phi_2 - \phi_1}{i_2 - i_1} = \frac{(45-30) \times 10^{-4}}{10-5} = \frac{15 \times 10^{-4}}{5}$$
$$= 3 \times 10^{-4} \text{ Wb/A}$$
$$\text{Mean inductance, } L = N\frac{d\phi}{di}$$
$$= 1000 \times 3 \times 10^{-4} = \mathbf{0.3 \text{ Henry}} \quad \textbf{Ans.}$$ ☜
$$\text{Induced emf} = L\frac{di}{dt}$$
$$= 0.3 \times \frac{10-5}{0.1} = \frac{0.3 \times 5}{0.1} = \mathbf{15 \text{ V}} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 3.31 *Evaluate the loop inductance per km of a single phase transmission circuit comprising two parallel conductor spaced 1 m apart and with diameters 0.5 cm and 0.8 cm respectively.*

☺ ***Solution:***

$$d = 1 \text{ m}$$
$$a = \frac{0.5}{2} = 0.25 \times 10^{-2} \text{ m}$$
$$b = \frac{0.8}{2} = 0.4 \times 10^{-2} \text{ m}$$
$$\mu_r = 1$$
$$\text{Inductance of transmission lines, } L = \frac{\mu_0}{4\pi}\left[\mu_r + 2\,ln\,\frac{d^2}{a\,b}\right]$$
$$= \frac{4\pi \times 10^{-7}}{4\pi}\left[1 + 2\,ln\left(\frac{10^4}{0.25 \times 0.4}\right)\right]$$
$$= 10^{-7}\,[1 + 23.026]$$
$$= 24.026 \times 10^{-7} \text{ H/m}$$
$$= 24.026 \times 10^{-4} \text{ H/km}$$
$$= \mathbf{2.4026 \text{ mH/km}} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 3.32 *Two coils of self-inductances of 0.5 H and 0.8 H with negligible resistance are connected in series. If their mutual inductance is 0.2 H, determine the effective inductance of the combination.*

☺ ***Solution:***

$$L_1 = 0.5\ \text{H}, \quad L_2 = 0.8\ \text{H}, \quad M = 0.2\ \text{H}$$

$$L_{eff} = L_1 + L_2 \pm 2\,M$$

$$L_{eff}\,(\text{aiding}) = 0.5 + 0.8 + 2 \times 0.2$$

$$= 1.7\ \text{H}$$

$$L_{eff}\,(\text{opposing}) = 0.5 + 0.8 - 2 \times 0.2 = \mathbf{0.9\ H}$$ **Ans.** ☜ ☺☺

Example 3.33 *Two inductance coils having self inductance of 4 H and 6 H are connected in parallel. The inductance of mutual coupling is 3 H. If the coupling between the coils is opposing the flux in each other and they are both across a common pair of terminals determine the equivalent inductance.*

☺ ***Solution:***

$$L_1 = 4\ \text{H}, \quad L_2 = 6\ \text{H}, \quad M = 3\ \text{H}$$

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2\,M} = \frac{24 - 9}{10 + 6}$$

$$= 0.9375\ \text{H} = \mathbf{937.5\ mH}$$ **Ans.** ☜ ☺☺

Example 3.34 *A square coil of 10 cm side with 100 turns is rotated at a uniform speed of 1000 rpm, about an axis at right angles to a magnetic field of density 0.5 Tesla. Calculate the instantaneous emf induced when the plane of the coil is (i) at 90° to the field (ii) in the plane of the field.*

☺ ***Solution:***

$$l = 10 \times 10^{-2} = 0.1\ \text{m}$$

$$N = 100$$

$$B = 0.5\ \text{Tesla}$$

$$\text{Speed} = 1000\ \text{rpm} = \frac{1000}{60}\ \text{rps}$$

$$\text{Velocity} = \pi\, l \times \text{speed}$$

$$= \frac{22}{7} \times 0.1 \times \frac{1000}{60} = 5.236\ \text{m/s}$$

$$\text{Induced emf per turn} = 2\,B\,v\,l \sin\theta \quad (\text{for 2 sides})$$

$$\text{Induced emf} = 2\,N\,B\,v\,l \sin\theta$$

$$= 2 \times 100 \times 0.5 \times 5.236 \times 0.1 \sin\theta$$

$$= 52.36 \sin\theta$$

(i) $\theta = 90°$

$$\text{emf} = 52.36 \sin 90° = \mathbf{52.36\ V}$$ **Ans.** ☜

(ii) $\theta = 0°$ (in the plane of the field)

$$\text{emf} = 52.36 \sin 0° = \mathbf{0\ V}$$ **Ans.** ☜ ☺☺

Example 3.35 *A cast steel (relative permeability 1000) ring of 0.2 m mean diameter and 3 sq.cm in cross-section has a winding of 200 turns. Find the current required to produce a flux of 375 micro webers.*

☺ ***Solution:***

$$N = 200$$

$$\mu_r = 1000$$

$$R = \frac{0.2}{2} = 0.1\ \text{m}$$

$$\phi = 375 \times 10^{-6}\ \text{Wb}$$

$$A = 3 \times 10^{-4}\ \text{m}^2$$

$$\text{Flux produced by toroid } \phi = B\,A$$

$$= \frac{\mu_0 \mu_r N I}{2\pi R}\, A$$

$$\therefore I = \frac{2\pi R \phi}{\mu_0 \mu_r N A}$$

$$= \frac{2\pi \times 0.1 \times 375 \times 10^{-6}}{4\pi \times 10^{-7} \times 1000 \times 200 \times 3 \times 10^{-4}}$$

$$\mathbf{I = 3.125\ A}$$ **Ans.** ☜ ☺☺

Example 3.36 *The combined inductance of two coils connected in series is 0.6 H or 0.1 H depending on the relative directions of the currents in the coils. If one of the coils when is isolated has a self-inductance of 0.2 H, calculate (i) the mutual inductance and (ii) the coupling coefficient.*

☺ **Solution:**

$$L_1 + L_2 + 2M = 0.6\ \text{H}$$

$$L_1 + L_2 - 2M = 0.1\ \text{H}$$

$$\text{But} \quad L_1 = 0.2\ \text{H}$$

$$L_2 + 2M = 0.4 \qquad \dots (1)$$

$$L_2 - 2M = -0.1 \qquad \dots (2)$$

Adding equations (1) and (2),

$$2L_2 = 0.3$$

$$L_2 = 0.15\ \text{H}$$

Subtracting equation (2) from (1),

$$4M = 0.5$$

$$\mathbf{M = 0.125\ H}$$ **Ans.** ☜

$$\text{Mutual inductance, } M = K\sqrt{L_1 L_2}$$

$$\text{Coefficient of coupling, } K = \frac{M}{\sqrt{L_1 L_2}} = \frac{0.125}{\sqrt{0.2 \times 0.15}}$$

$$\mathbf{K = 0.7217}$$ **Ans.** ☜

☺☺

Example 3.37 *Evaluate the inductance of a solenoid of 2500 turns wound uniformly over a length of 0.5 m on a cylindrical paper tube 4 cm in diameter. The medium is air.*

☺ **Solution:**

$$N = 2500$$

$$l = 0.5\ \text{m}$$

$$d = 4\ \text{cm} = 4 \times 10^{-2}\ \text{m}$$

$$A = \frac{\pi d^2}{4}$$

$$= \frac{\pi \times 4 \times 4 \times 10^{-4}}{4} = 12.566 \times 10^{-4}\ \text{m}^2$$

$$\text{Inductance of solenoid, } L = \frac{\mu_0 N^2 A}{l}$$

$$= \frac{4\pi \times 10^{-7} \times (2500)^2 \times 12.566 \times 10^{-4}}{0.5}$$

$$\mathbf{L = 19.7386\ mH}$$ **Ans.** ☜

☺☺

Example 3.38 *Two coils are wound on a common magnetic circuit of 50 cm^2 in section and mean length of 100 cm. One coil has 80 turns and the other 700. Calculate the mutual inductance if μ_r of the iron path is 1500.*

☺ **Solution:**

$$N_1 = 80$$

$$N_2 = 700$$

$$l = 1\ \text{m}$$

$$A = 50 \times 10^{-4}\ \text{m}$$

$$\mu_r = 1500$$

$$\text{Mutual inductance, } M = \frac{N_1 N_2 \mu_0 \mu_r A}{l}$$

$$= \frac{80 \times 700 \times 4\pi \times 10^{-7} \times 1500 \times 50 \times 10^{-4}}{1}$$

$$= \mathbf{0.528\ H}$$ **Ans.** ☜

☺☺

Example 3.39 *A solenoid consists of 1500 turns of wire wound on a length of 60 cm. A search coil of 500 turns enclosing a mean area of 20 cm^2 is placed centrally in the solenoid. Find (a) the mutual inductance of the arrangement, (b) the electromotive force induced in the search coil when the current in the solenoid is changing uniformly at the rate of 250 A/sec.*

☺ **Solution:**

$$N_1 = 1500$$

$$l = 0.6\ \text{m}$$

$$N_2 = 500$$

$$A = 20 \times 10^{-4}\ \text{m}^2$$

$$\frac{di}{dt} = 250\ \text{A/s}$$

(*a*)

$$\text{Mutual inductance, } M = \frac{N_1 N_2 \mu_0 A}{l}$$

$$= \frac{1500 \times 500 \times 4\pi \times 10^{-7} \times 20 \times 10^{-4}}{0.6}$$

$$= \mathbf{3.1416\ mH}$$ **Ans.** ☜

(*b*) The induced emf in search coil due to change in the current in solenoid

$$v = M\frac{di}{dt} = 3.1416 \times 10^{-3} \times 250$$

$$= \mathbf{0.7854\ V}$$ **Ans.** ☜

☺☺

Example 3.40 *Find the energy stored in the solenoid having 50 cm long and 5 cm in diameter and is wound with 2000 turns of wire, carrying a current of 10 A.*

☺ ***Solution:***

$$N = 2000$$

$$d = 5 \text{ cm}$$

$$A = \pi\left(\frac{5}{2}\times 10^{-2}\right)^2 = 19.635\times 10^{-4}\text{ m}^2$$

$$l = 0.5\text{ m}$$

$$I = 10\text{ A}$$

$$\text{Inductance of solenoid, } L = \frac{\mu_0 N^2 A}{l}$$

$$= \frac{4\pi\times 10^{-7}\times(2000)^2\times 19.635\times 10^{-4}}{0.5}$$

$$= 0.01974\text{ H}$$

$$\text{Energy stored} = \frac{1}{2}LI^2 = \frac{1}{2}\times 0.01974\times 10^2$$

$$= \textbf{0.987 Joules}$$ **Ans.** ☜ ☺☺

Example 3.41 *A toroid is wound with 300 turns on an ebonite ring having a cross-sectional area of 4 cm² and a mean circumference of 35 cm. Calculate (i) the inductance of coil, (ii) induced emf when the current is reduced at the rate of 200 A/sec, (iii) if the toroid has a second winding of 80 turns wound over the ebonite ring and inside the first winding of 300 turns, calculate the mutual inductance.*

☺ ***Solution:***

$$N = 300,\quad A = 4\times 10^{-4}\text{ m}^2$$

$$l = 2\pi R = 35\text{ cm} = 35\times 10^{-2}\text{ m}$$

$$\frac{di}{dt} = 200\text{ A/sec}$$

(*i*) $$\text{Inductance of a toroid} = \frac{\mu N^2 A}{l}\text{ H/m}$$

$$= \frac{4\pi\times 10^{-7}(300)^2\times 4\times 10^{-4}}{35\times 10^{-2}}$$

$$= \textbf{0.1293 mH/m}$$ **Ans.** ☜

(*ii*) $$\text{Induced emf, } e = L\frac{di}{dt} = 0.1293\times 10^{-3}\times 200$$

$$= \textbf{25.85 mV}$$ **Ans.** ☜

(*iii*) For second coil, $$N_2 = 80,\quad A = 4\times 10^{-4}\text{ m}^2$$

$$l = 35\times 10^{-2}\text{ m}$$

$$\text{Inductance of second coil, } L_2 = \frac{\mu N^2 A}{l}\text{ H/m}$$

$$= \frac{4\pi\times 10^{-7}(80)^2\times 4\times 10^{-4}}{35\times 10^{-2}}$$

$$L_2 = 9.191\times 10^{-6}\text{ H/m}$$

Mutual inductance between two coils

$$M = K\sqrt{L_1 L_2}$$

$$= \sqrt{0.1293\times 10^{-3}\times 9.191\times 10^{-6}}$$

$$= \mathbf{3.44\times 10^{-5}\ H/m}$$ **Ans.** ☜ ☺☺

Example 3.42 *Two coils are connected in series and their total inductance is 4.4 mH. When one coil is reversed, the total inductance is 1.6 mH. All the flux due to the first coil links the second coil, but only 40% of the flux due to second coil links the first coil. Find the self inductance of each coil and their mutual inductance.*

☺ ***Solution:*** Let ϕ_{12} be the flux linked with second coil due to ϕ_1. Let ϕ_{21} be the flux linked with first coil due to ϕ_2.

Given: $$\phi_{12} = \phi_1,\quad \phi_{21} = \frac{40}{100}\phi_2 = 0.4\,\phi_2$$

Let L_1, L_2 be the respective inductances of coils 1 and 2.

$$K_1 = \frac{\phi_{12}}{\phi_1} = 1 \text{ and } K_2 = \frac{\phi_{21}}{\phi_2} = 0.4$$

Inductance in series is given by

$$L_1 + L_2 + M_1 + M_2 = 4.4\times 10^{-3}\text{ H} \quad \ldots(1)\ [\because \text{Given}]$$

$$L_1 + L_2 - M_1 - M_2 = 1.6\times 10^{-3}\text{ H} \quad \ldots(2)\ [\because \text{Given}]$$

where, M_1 is the mutual inductance between coil 1 and 2 due to ϕ_1

M_2 is the mutual inductance between coil 1 and 2 due to ϕ_2

Adding equations (1) and (2),

$$2(L_1 + L_2) = 6 \times 10^{-3} \text{ H} \quad \ldots (3)$$

$$L_1 + L_2 = 3 \times 10^{-3} \text{ H} \quad \ldots (4)$$

Subtracting equation (2) from (1),

$$2(M_1 + M_2) = 2.8 \times 10^{-3} \text{ H} \quad \ldots (5)$$

$$M_1 + M_2 = 1.4 \times 10^{-3} \text{ H} \quad \ldots (6)$$

$$\text{But} \quad M = K\sqrt{L_1 L_2}$$

Substituting the above equation in equation (6),

$$K_1\sqrt{L_1 L_2} + K_2\sqrt{L_1 L_2} = 1.4 \times 10^{-3} \text{ H}$$

$$\sqrt{L_1 L_2}\,(K_1 + K_2) = 1.4 \times 10^{-3} \text{ H}$$

$$\sqrt{L_1 L_2}\,(1 + 0.4) = 1.4 \times 10^{-3} \text{ H} \quad [\because K_1 = 1 \text{ and } K_2 = 0.4]$$

$$\sqrt{L_1 L_2} = 1 \times 10^{-3} \text{ H}$$

Squaring on both sides,

$$L_1 L_2 = 1 \times 10^{-6} \text{ H} \quad \ldots (7)$$

From equation (7),

$$L_2 = \frac{1 \times 10^{-6}}{L_1} \text{ H} \quad \ldots (8)$$

Substituting equation (8) in equation (4),

$$L_1 + \frac{1 \times 10^{-6}}{L_1} = 3 \times 10^{-3} \quad \ldots (9)$$

Rearranging,

$$L_1^2 - 3 \times 10^{-3} L_1 + 1 \times 10^{-6} = 0 \quad \ldots (10)$$

Solving for quadratic equation (10),

$$L_1 = \frac{3 \times 10^{-3} \pm \sqrt{9 \times 10^{-6} - 4 \times 10^{-6}}}{2}$$

$$L_1 = \frac{3 \times 10^{-3} \pm 2.23 \times 10^{-3}}{2}$$

$$L_1 = 2.618 \text{ mH (or) } L_1 = 0.382 \text{ mH}$$

Assume $L_1 = 2.618$ mH, then

$$L_2 = \frac{1 \times 10^{-6}}{L_1} = \frac{1 \times 10^{-6}}{2.618 \times 10^{-3}} = 0.382 \text{ mH}$$

Mutual inductance:

$$M_1 = K_1\sqrt{L_1 L_2} = 1\sqrt{0.382 \times 10^{-3} \times 2.618 \times 10^{-3}} = 1 \text{ mH}$$

$$M_2 = K_2\sqrt{L_1 L_2} = 0.4\sqrt{0.382 \times 10^{-3} \times 2.618 \times 10^{-3}} = 0.4 \text{ mH}$$

Inductances	**Value in milli henry**
L_1	2.618
L_2	0.382
M_1	1
M_2	0.4

☺☺

Example 3.43 *A coil is continuously wound over a cylindrical soft iron core and the two ends are drawn out together. If the radius of the soft-iron core that is circular is 20 mm and the external radius of the core cylinder is 4 mm, determine the self inductance of the arrangement. The total number of turns due to the winding are 2500.*

☺ ***Solution:***

Given: $r = 4 \times 10^{-3}$ m, $N = 2500$, $R_m = 20 \times 10^{-3}$ m

$$L = \frac{\mu N^2 r^2}{2 R_m} = \frac{4\pi \times 10^{-7} (2500)^2 \times (4 \times 10^{-3})^2}{2 \times (20 \times 10^{-3})}$$

$$L = \pi \text{ mH}$$

Hence, inductance of the toroid is $\pi \times 10^{-3}$ **H. Ans.** ☜

☺☺

Example 3.44 *Determine the self inductance of a solenoid coil that has 2500 turns, length of 0.5 m and a cylindrical air core of radius 2 cm.*

☺ ***Solution:***

Given: $N = 2500$, $l = 0.5$ m, $r = 2 \times 10^{-2}$ m

$$L = \frac{N^2 \mu A}{l} = \frac{(2500)^2 \times 4\pi \times 10^{-7} \times \pi \times (2 \times 10^{-2})^2}{0.5}$$

$$L = \mathbf{19.739 \text{ mH}}$$ **Ans.** ☜

☺☺

Example 3.45 *A circular loop of wire is placed in a uniform magnetic field of flux density 0.5 wb/m^2. The wire has 200 turns and frequency of rotation of 1000 revolution/minute. If the radius of the coil is 0.2 m, determine,*

(a) the induced emf, when the plane of the coil is perpendicular to the field,

(b) the induced emf, when the plane of the coil is 60° to the flux lines.

☺ **Solution:**

$$r = 0.2 \text{ m}$$

$$N = 200$$

$$f = 1000 \text{ rpm} = \frac{1000}{60} \text{ rps}$$

$$\omega = \frac{2\pi \times 1000}{60} \text{ rad/s}$$

$$B = 0.5 \text{ wb/m}^2$$

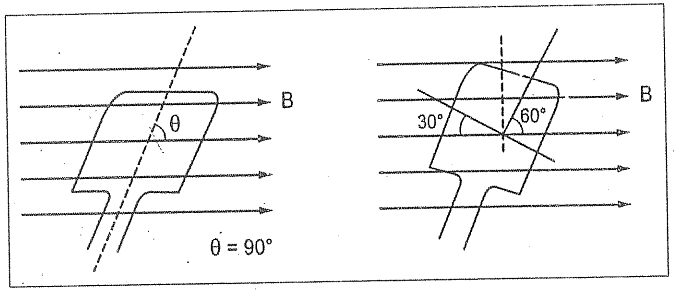


Fig. 3.28.

When the plane of the coil is perpendicular to the field, the induced emf

$$e = NBA\,\omega \sin\theta$$

$$= 200 \times 0.5 \times \pi \times (0.2)^2 \times \frac{2\pi \times 1000}{60} \sin 90°$$

$$= \mathbf{1316 \text{ volts}} \quad \textbf{Ans.}$$

When plane is at 60° to the field, then the induced emf is

$$e = 200 \times 0.5 \times \pi \times (0.2)^2 \times \frac{2\pi \times 1000}{60} \times \sin 60°$$

$$= \mathbf{1139.644 \text{ volts}} \quad \textbf{Ans.}$$

☺☺

Example 3.46 *A coaxial cable has two conductors aligned radially with radius 1 mm of the core conductor and 7 mm of the outer conductor. If the insulator is in free space determine the self-inductance of the coaxial cable when the length is known to be 100 m.*

☺ **Solution:**

$$l = 100 \text{ m}, \quad a = 10^{-3} \text{ m}, \quad b = 7 \times 10^{-3} \text{ m}$$

$$L = \frac{\mu}{2\pi} \ln(b/a) \text{ H/m} = \frac{4\pi \times 10^{-7}}{2\pi} \ln(7/1)$$

$$= 389.182 \times 10^{-9} \text{ H/m}$$

$$\text{For length of } l = 100 \text{ m}$$

$$L = \mathbf{38.92 \times 10^{-6} \text{ H}} \quad \textit{Ans.}$$

☺☺

Example 3.47 *(a) A solenoid has length of 2 m and a coil of 4000 turns wound over a core of radius 5 cm. Determine the self-inductance of the solenoid of the core is just an air-core.*

(b) Two coils are having a perfect coupling between themselves. Both the coils are wound over a common core of relative permeability of 100 and radius 2 cm, length being 50 cm. Determine the inductance of mutual coupling, if the two coils have 100 and 400 turns.

☺ ☺**Solution:**

(a)

$$N = 4000 \text{ turns}, \quad l = 2 \text{ m}, \quad r = 5 \times 10^{-2} \text{ m}$$

$$L = \frac{\mu N^2 A}{l}$$

$$= \frac{4\pi \times 10^{-7} (4000)^2 \times [\pi \times (5 \times 10^{-2})^2]}{2}$$

$$L = \mathbf{78.96 \text{ mH}} \quad \textbf{Ans.}$$

(b)

$$\mu_r = 100 \qquad r = 2 \times 10^{-2} \text{ m}$$

$$N_1 = 100 \qquad l = 0.5 \text{ m}$$

$$N_2 = 400 \qquad K = 1$$

$$L_1 = \frac{N_1^2 \mu A}{l}$$

$$= \frac{100^2 \times 4\pi \times 10^{-7} \times 100 (\pi \times 4 \times 10^{-4})}{0.5} = 3.158 \text{ mH}$$

$$L_2 = \frac{N_2^2 \mu A}{l}$$

$$= \frac{(400)^2 \times 4\pi \times 10^{-7} \times 100\,(\pi \times 4 \times 10^{-4})}{0.5} = 50.532 \text{ mH}$$

$$M = K\sqrt{L_1 L_2} = \sqrt{(3.158)(50.532)} \text{ mH}$$

$$= \mathbf{12.632 \text{ mH}} \quad \textbf{Ans.}$$ ☞

☺☺

Example 3.48 *Calculate the inductance of a solenoid of 200 turns wound on a cylindrical tube 6 cms diameter. The length of the tube is 60 cm, and the medium is air.*

☺ ***Solution:***

$N = 200$ turns

$r = .3 \text{ cm} = 0.03 \text{ m}$

$l = 60 \text{ cm} = 0.6 \text{ m}$

$\mu_r = 1$

$A = \pi r^2$

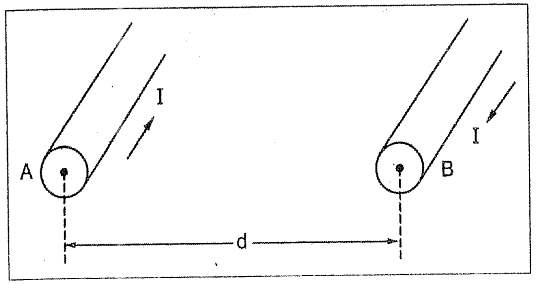
Inductance of the solenoid is given by

$$L = \frac{\mu_0 N^2 A}{l} = \frac{4\pi \times 10^{-7} \times (200)^2 \times \pi \times (0.3)^2}{0.6}$$

$$\mathbf{L = 0.2368 \text{ mH}} \quad \textbf{Ans.}$$ ☞

☺☺

Example 3.49 *Calculate the internal and external inductance per unit length of a transmission line consisting of two long parallel conducting wires of radius 'a' that carry currents in opposite directions. The axes of the wires are separated by a distance 'd', which is much larger than 'a'.*



☺ ***Solution:*** Consider a transmission line consisting of two long parallel conducting wires A and B of radius 'a' that carry currents in opposite directions. The two conductors are separated by a distance 'd' which is much larger than the radius 'a'.

The internal flux linkage of the wires 1 and 2,

$$\phi = \phi_A + \phi_B = \frac{\mu_0 \mu_r I}{8\pi} + \frac{\mu_0 \mu_r I}{8\pi} = \frac{\mu_0 \mu_r I}{4\pi}$$

Internal inductance of transmission line

$$L_{in} = \frac{\phi}{I}$$

$$\mathbf{L_{in}} = \frac{\mu_0 \mu_r}{4\pi} \quad \textbf{Ans.}$$ ☞

Magnetic flux density at any point with a distance 'x' due to the conductor A is

$$B_A = \frac{\mu_0 I}{2\pi x}$$

Magnetic flux density due to the conductor B is

$$B_B = \frac{\mu_0 I}{2\pi (d-x)}$$

The total flux linkage, $\phi = \int_a^{d-a} (B_A + B_B)\, dx$

$$= \frac{\mu_0 I}{2\pi} \int_a^{d-a} \left(\frac{1}{x} + \frac{1}{d-x}\right) dx$$

$$= \frac{\mu_0 I}{2\pi}\left[\log\frac{d-a}{a} - \log\frac{a}{d-a}\right]$$

$$= \frac{\mu_0 I}{2\pi}\, 2\log\frac{d-a}{a}$$

$$\phi = \frac{\mu_0 I}{\pi} \log\frac{d-a}{a}$$

External inductance of transmission line

$$\mathbf{L_{ext}} = \frac{\phi}{I} = \frac{\mu_0}{\pi} \log\frac{\boldsymbol{d-a}}{\boldsymbol{a}} \quad \textbf{Ans.}$$ ☞

☺☺

Example 3.50 *A circular loop of N turns of conducting wire lies in the xy plane with its centre at the origin of a magnetic field specified by* $B = \bar{a}_z B_0 \cos\left(\frac{\pi r}{2b}\right) \sin \omega t$, *where b is the radius of the loop and ω is the angular frequency, Find the emf induced in the loop.*

☺ ***Solution:*** The flux linkage with the loop

$$\phi = \int_s B \cdot ds$$

$$= \int_0^b \left[B_0 \cos \frac{\pi r}{2b} \sin \omega t \right] \bar{a}_z \cdot 2\pi r \, dr$$

$$= B_0 2\pi \sin \omega t \int_0^b \cos\left(\frac{\pi r}{2b}\right) r \, dr$$

$$\text{Let } x = \frac{\pi r}{2b}, \quad r = \frac{2b}{\pi} x \quad \text{and} \quad dx = \frac{\pi}{2b} \cdot dr$$

$$\phi = 2\pi B_0 \sin \omega t \left(\frac{2b}{\pi}\right)^2 \int_0^{\pi/2} x \cos x \, dx$$

$$= 2\pi B_0 \sin \omega t \left(\frac{2b}{\pi}\right)^2 \left(\frac{\pi}{2} - 1\right)$$

$$\phi = \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t$$

Since there are N number of turns, the total flux is N ϕ.

According to Faraday's law,

$$\text{emf} = -N \frac{d\phi}{dt}$$

$$= -N \frac{d}{dt} \left[\frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t \right]$$

$$= -\frac{8N}{\pi} b^2 \left(\frac{\pi}{2} - 1\right) B_0 \omega \cos \omega t$$ **volts Ans.** ☞

☺☺

TWO MARK QUESTIONS AND ANSWERS

1. *State Biot-Savart law.*

It states that the magnetic flux density at any point due to current element is proportional to the current element and sine of the angle between the elemental length and line joining and inversely proportional to the square of the distance between them.

$$dB = \frac{\mu_0 I \, dl}{4\pi r^2} \sin \theta$$

2. *Define magnetic flux density.*

$$\text{Magnetic flux density } B = \frac{\text{Magnetic flux}}{\text{Area}}$$

$$B = \frac{\phi}{A} \text{ Webers/m}^2 \text{ (Tesla)}$$

3. *What is the relation between magnetic flux density and field intensity?*

$$\text{Magnetic flux density } B = \mu H$$

$$\text{where, } \mu = \mu_0 \mu_r$$

$$\text{Field intensity } H = \frac{B}{\mu}$$

4. *State Ampere's circuital law.*

Magnetic field intensity around a closed path is equal to the current enclosed by that path

$$\oint H \cdot dI = I$$

5. *List the applications of Ampere's circuital law.*

Application of Ampere's circuital law involves finding the total current enclosed by a closed path.

e.g., It is used to find the magnetic field intensity due to infinitely long straight conductor and coaxial cable.

6. *Define magnetic scalar potential.*

It is defined as dead quantity whose negative gradient gives the magnetic intensity if there is no current source present.

$$H = -\nabla V_m$$

where V_m is the magnetic scalar potential.

$$V_m = -\int H \cdot dl$$

7. ***Define magnetic vector potential.***

It is defined as that quantity whose curl gives the magnetic flux density

$$B = \nabla \times A$$

where A is the magnetic vector potential.

$$A = \frac{\mu}{4\pi} \iiint_v \frac{J}{r}\, dr \quad \text{web/m}$$

It is used to find the magnetic field due to current J.

8. ***Distinguish between magnetic scalar potential and magnetic vector potential.***

S.No.	Magnetic scalar potential	Magnetic vector potential
1.	It is defined as dead quantity whose negative gradient gives the magnetic intensity if there is no current source present.	It is defined as that quantity whose curl gives the magnetic flux density.
2.	$H = -\nabla V_m$	$B = \nabla \times A$

9. ***Define magnetic moment.***

Magnetic moment is defined as the maximum torque per magnetic induction (flux density).

$$m = \frac{T}{B}$$

$$m = IA \quad \text{where A is area.}$$

10. ***Give four similarities between electrostatic field and magnetic field.***

Electrostatic field	Magnetic field
Electric field intensity E volts/m	Magnetic field intensity H A/m
Electric flux density $D = \varepsilon E$ C/m^2	Magnetic flux density $B = \mu H$ web/m^2
Energy stored is $\frac{1}{2} C V^2$ J	Energy stored is $\frac{1}{2} L I^2$ J
Charges are at rest	Charges are in motion

11. ***Write down the expression for magnetic field at the centre of the circular coil.***

$$H = \frac{I}{2a}$$

12. ***What is the magnetic field intensity due to toroid and solenoid?***

(a) Toroid $H = \dfrac{NI}{2\pi r}$

(b) Solenoid $H = \dfrac{NI}{l}$

13. ***A circular coil of radius 2 m carries a current of 4 A. What is the value of magnetic field intensity at the centre?***

$$I = 4\,A; \qquad a = 2\,m$$

$$H = \frac{I}{2a} = \frac{4}{2 \times 2} = 1 \text{ A/m}$$

14. ***Write the force equation on a current element.***

The force on a current element $I\,dl$ is given by

$$dF = I\,dl \times B$$

15. ***Give torque equation on closed circuits.***

The torque on closed circuit in a magnetic field is

$$T = BIA \sin\theta$$

$$T = mB \sin\theta \quad \text{where } m \text{ is magnetic moment.}$$

In vector form, $T = m \times B$

16. ***Give an equation of torque on a solenoid.***

Torque on a solenoid in a magnetic field is

$$T = \frac{n}{2} \cdot 2IAB = nBIA$$

$$= mB \quad \text{where } m = nIA$$

17. ***What is solenoid?***

An inductor coil consisting of multiple turns of wire wound in a helical geometry around a cylindrical core is called as solenoid.

18. *Write down the equation for general, integral and point form of Ampere's law.*

General form: $\oint H\,dl = I$

Integral form: $\oint H\,dl = \iint_S J \cdot ds$

Point form: $\nabla \times H = J$

19. *A solenoid with a radius of 2 cm is wound with 20 turns per cm length and carries 10 mA. Find H at the centre if the total length is 10 cm.*

$$N = nl = 20 \times 10 = 200 \text{ turns}$$

$$l = 10 \times 10^{-2}$$

$$I = 10 \times 10^{-3}$$

$$H = \frac{NI}{l} = \frac{200 \times 10 \times 10^{-3}}{10 \times 10^{-2}} = 20 \text{ AT/m}$$

20. *Find the maximum torque on an 100 turn rectangular coil, 0.2 m by 0.3 m, carrying a current of 2A in the field of flux density 5 web/m².*

$$N = 100$$

$$A = 0.2 \times 0.3 = 0.06 \text{ m}^2$$

$$I = 2 \text{ A}$$

$$B = 5 \text{ web/m}^2$$

$$T_{max} = NIAB = 100 \times 2 \times 0.06 \times 5$$

$$= 60 \text{ Newton-metre}$$

21. *A loop with magnetic dipole moment $8 \times 10^{-3}\,\vec{a_z}$ Am², lies in a uniform magnetic field $\vec{B} = 0.2\,\vec{a_x} + 0.4\,\vec{a_z}$ Wb/m². Calculate the torque.*

$$m = 8 \times 10^{-3}\,\vec{a_z}$$

$$\vec{B} = 0.2\,\vec{a_x} + 0.4\,\vec{a_z}$$

$$\text{Torque } T = m \times \vec{B} = 8 \times 10^{-3}\,\vec{a_z}\,(0.2\,\vec{a_x} + 0.4\,\vec{a_z})$$

$$= 16 \times 10^{-4}\,\vec{a_y} + 32 \times 10^{-4}\,(0)$$

$$= 16 \times 10^{-4}\,\vec{a_y} \text{ N-m}$$

22. *Write the expression for the force on a test charge – q that moves with velocity $\bar{v}$ in a magnetic field of flux density $\bar{B}$. Find its direction with respect to $\bar{v}$ and $\bar{B}$.*

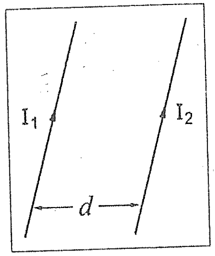
The force on a test charge – q that moves with velocity $\bar{v}$ in a magnetic field of flux density $\bar{B}$ is given by

$$F = q\,\bar{v} \times \bar{B} = q\,\bar{v}\,\bar{B}\,\sin\theta$$

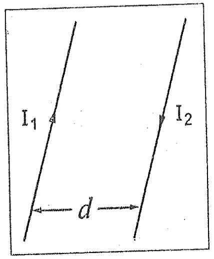
$$= |\,q\,|\,\bar{v}\,\bar{B}$$

The direction of the force is perpendicular to both $\bar{v}$ and $\bar{B}$ ($\theta = 90°$).

23. *Two current carrying wires carry I_1 and I_2 in (a) the same direction, (b) in opposite direction. Determine the force between them assuming a separation 'd'.*



(*a*) Force of attraction $F = \dfrac{\mu_0 I_1 I_2}{2\pi d}$



(*b*) Force of repulsion $F = \dfrac{\mu_0 I_1 I_2}{2\pi d}$

24. Determine the force per unit length between two long parallel wires separated by 5 cm in air and carrying currents of 40 A in the same direction.

$$\text{Force/length} = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{4\pi \times 10^{-7} \times 40 \times 40}{2\pi \times 5 \times 10^{-2}}$$

$$= 6.4 \times 10^{-3} \text{ N/m}$$

25. Define self-inductance.

The self-induction of a coil is defined as the ratio of total magnetic flux linkage with the circuit to the current through the coil.

where, ϕ is magnetic flux

N is number of turns of coil

i is the current

$$L = \frac{\phi}{i}$$

26. Define mutual inductance.

The mutual inductance between two coils is defined as the ratio of induced magnetic flux linkage in one coil to the current through the other coil.

$$M = \frac{N_2 \phi_{12}}{i_1}$$

where, N_2 is number of turns in coil 2

ϕ_{12} is magnetic flux linking with coil 2 due to coil 1

i_1 is the current through coil 1

27. Define coupling coefficient.

The fraction of the total flux produced by one coil linking the second coil is called the coefficient of coupling (K).

$$K = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

ϕ_1 is the flux produced by coil 1

ϕ_{12} is flux linking with coil 2

$$K \leq 1$$

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

28. How do mutual inductance between two coils is related to their self-inductances?

$$M = K\sqrt{L_1 L_2}$$

where, K is coupling coefficient

L_1 is self-inductance of coil 1

L_2 is self-inductance of coil 2

M is mutual inductance

29. What will be effective inductance, if two inductors are connected in (a) series and (b) parallel?

(*a*) For series $L = L_1 + L_2 \pm 2M$ + sign for aiding

(*b*) For parallel $L = \dfrac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}$ – sign for opposition

30. Two coils, having self-inductance of 1 mH and 2 mH and mutual inductance of 0.5 mH are connected in series. Find the effective inductance of series combination.

$$L_1 = 1 \text{ mH}$$

$$L_2 = 2 \text{ mH}$$

$$M = 0.5 \text{ mH}$$

(*i*) $L = L_1 + L_2 - 2M$ (opposition)

$$= 1 + 2 - 1 = 2 \text{ mH}$$

(*ii*) $L = L_1 + L_2 + 2M$ (aiding)

$$= 1 + 2 + 1 = 4 \text{ mH}$$

31. What is the mutual inductance of two inductively tightly coupled coils with self-inductance of 25 mH and 100 mH?

The magnetic lines of force produced by a magnet are called magnetic flux.

$$L_1 = 25 \text{ mH}, \quad K = 1$$

$$L_2 = 100 \text{ mH}$$

$$M = K\sqrt{L_1 L_2} = \sqrt{25 \times 100} = 50 \text{ mH}$$

32. *Give the expression for inductance of a solenoid.*

$$L = \frac{\mu_0 N^2 A}{l}$$

where, N is number of turns
A is area of cross-section
l is length of solenoid
μ_0 is free space permeability

33. *Give the expression for inductance of a toroid.*

$$L = \frac{\mu_0 N^2 A}{2\pi R} = \frac{\mu_0 N^2 r^2}{2 R} \qquad [\because A = \pi r^2]$$

where, N is number of turns
r is radius of the coil
R is radius of toroid
μ_0 is free space permeability

34. *What is the loop inductance per kilometre of a single phase overhead line with r = 1 cm and D = 80 cm.*

$$L = \frac{\mu_0}{2\pi} \ln(D/r) \quad \text{H/m}$$

$$= \frac{4\pi \times 10^{-7}}{2\pi} \ln\left(\frac{80}{1}\right) = 0.876 \text{ micro H/km}$$

35. *Give the expression for inductance per unit length of a coaxial transmission line.*

$$L = \frac{\mu_0}{2\pi} \ln(b/a) \quad \text{H/m}$$

where, a is the radius of inner conductor
b is the radius of outer conductor

36. *What is torque on a solenoid?*

Torque T = 2 B I A
where, B is the flux density
I is the current through solenoid
A is the area of loop

If solenoid has N number of turns, the maximum torque

$$T_{max} = \frac{N}{2} \cdot 2 B I A$$

$$T = N B I A \quad \text{Newton-metre}$$

37. *Distinguish between solenoid and toroid.*

Solenoid is a cylindrically shaped coil consisting of a large number of closely spaced turns of insulated wire wound usually on a non-magnetic frame.

If a long, slender solenoid is bent into the form of a ring and thereby closed on itself, it becomes toroid.

38. *What is the expression for energy stored in magnetic field?*

$$W = \frac{1}{2} L I^2$$

where, L is the inductance
I is the current

39. *What is energy density in the magnetic field?*

$$\text{Energy density, } W = \frac{1}{2} B H = \frac{1}{2} \mu H^2 = \frac{1}{2} \frac{B^2}{\mu}$$

40. *An inductive coil of 10 mH is carrying a current of 10 A. What is the energy stored in the magnetic field?*

$$L = 10 \times 10^{-3} \text{ H}$$

$$I = 10 \text{ A}$$

$$\text{Stored energy, } W = \frac{1}{2} L I^2$$

$$= \frac{1}{2} \times 10 \times 10^{-3} \times (10)^2 = 0.5 \text{ Joules}$$

41. *Compare the energy stored in inductor and capacitor.*

For magnetic field, Energy stored in inductor $W = \frac{1}{2} L I^2$

For electric field, Energy stored in capacitor $W = \frac{1}{2} C V^2$

42. *Write the expression for the torque experienced by a current carrying loop of area A m^2 situated in a magnetic field of B Wb/m^2.*

$$\text{Torque } T = B\,I\,A$$

where, I is the current

43. *What is meant by magnetisation?*

Magnetisation is defined as the magnetic dipole moment per unit volume.

If there are n magnetic dipoles per unit volume Δv, then magnetization

$$M = \underset{\Delta v \to 0}{\text{Lt}} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} m_i$$

44. *Classify the magnetic materials.*

Magnetic materials are classified into three groups.

(*i*) Diamagnetic
(*ii*) Paramagnetic
(*iii*) Ferromagnetic

45. *Distinguish between diamagnetic, paramagnetic and ferromagnetic materials.*

Diamagnetic: In diamagnetic materials, magnetization is opposed to the applied field. It has weak magnetic field.

Paramagnetic: In paramagnetic materials, magnetization is in the same direction as the field. It has weak magnetic field.

Ferromagnetic: In ferromagnetic materials, magnetization is in the same direction as the field. It has strong magnetic field.

46. *State the boundary conditions at the interface between two perfect dielectrics.*

(*i*) The tangential component of electric field is continuous

i.e., $E_{t1} = E_{t2}$.

(*ii*) The normal component of electric flux density is continuous

i.e., $D_{n1} = D_{n2}$

47. *Write Lorentz force equation and mention its applications.*

Lorentz force equation is given by

$$F = m\,a = Q\,(E + V \times B)$$

Application: Used to determine the motion of charged particle in a combined electric and magnetic field.

48. *Why flux density is called as solenoidal vector in a closed surface?*

By Gauss's law for magnetic field, the flux density in a closed surface is equal to zero.

i.e., $\nabla \cdot B = 0$

So, Flux density (B) is called as solenoidal vector in a closed surface.

SUMMARY

Fixed density

$$B = \frac{\text{Flux } (\phi)}{\text{Area } (A)} \text{ web/m}^2$$

$$B = \mu H$$

$$\nabla \cdot B = 0$$

Biot-Savart's Law

$$dB = \frac{\mu\, I\, dl \sin\theta}{4\pi r^2}$$

$$dH = \frac{I\, dl \sin\theta}{4\pi r^2}$$

Ampere's circuital Law

$$\oint H \cdot dl = I$$

Magnetic field at any point

(a) due to infinitely straight current carrying conductor

$$B = \frac{\mu_0 I}{2\pi d} \text{ web/m}^2$$

$$H = \frac{I}{2\pi d} \text{ A/m}$$

(b) due to finite length current carrying conductor

$$B = \frac{\mu_0 I}{4\pi d} [\cos\theta_1 + \cos\theta_2]$$

(i) along the axis of the solenoid

$$B = \frac{\mu_0 N I}{2\,l} [\cos\theta_1 + \cos\theta_2]$$

(ii) at the mid-point of solenoid

$$B = \frac{\mu_0 N I}{2\sqrt{(l/2)^2 + a^2}}$$

$$H = \frac{N I}{2\sqrt{(l/2)^2 + a^2}}$$

(iii) at the end of the axis of solenoid

$$B = \frac{\mu_0 N I}{2\sqrt{l^2 + a^2}}$$

(c) at the centre of Toroid coil

$$B = \frac{\mu_0 N I}{2\pi r}$$

$$H = \frac{N I}{2\pi r}$$

(d) along the axis of the circular coil

$$B = \frac{\mu_0 a^2 I}{2(a^2 + d^2)^{3/2}}$$

$$H = \frac{a^2 I}{2(a^2 + d^2)^{3/2}}$$

(i) at the centre of coil $B = \frac{\mu_0 I}{2a}$

$$H = \frac{I}{2a}$$

Self inductance $L = N\frac{d\phi}{di}$

Mutual inductance $M = N_2 \frac{d\phi_{12}}{di_1} = N_1 \frac{d\phi_{21}}{di_2}$

Coefficient of coupling

$$K = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

$$K = \frac{M}{\sqrt{L_1 L_2}}$$

$$M = K\sqrt{L_1 L_2}$$

Equivalent inductance

(a) for series connection

$L = L_1 + L_2 + 2M$ (aiding)

$L = L_1 + L_2 - 2M$ (opposing)

(b) for parallel connection

$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$ (aiding)

$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$ (opposing)

Inductance of solenoid

$$L = \frac{\mu_0 N^2 A}{l}$$

Inductance of toroid

$$L = \frac{\mu_0 N^2 r^2}{2R}$$

Inductance of coaxial cable for unit length

$$L = \frac{\mu_0 I}{2\pi} ln\left(\frac{b}{a}\right)$$

Inductance of coaxial cable with solid inner conductor

$$L = \frac{\mu_0}{4\pi}\left[\frac{1}{2}\mu_r + 2\, ln\left(\frac{b}{a}\right)\right] \text{ H/m}$$

Inductance of two transmission lines per unit length

$$L = \frac{\mu_0}{4\pi}\left[\mu_r + 2\,ln\,\frac{d^2}{a\,b}\right]$$

Energy stored in magnetic field

$$W = \frac{1}{2}\,L\,I^2$$

Magnetic energy density $W = \frac{1}{2}B\,H$

Force between two parallel conductors

$$F = \frac{\mu_0\,I_1\,I_2\,l}{2\pi\,d}$$

$$\text{Torque}\quad T = B\,I\,A$$

$$= m \times B \quad \text{(vector form)}$$

Magnetic moment $m = I\,A$

Magnetic scalar potential

$$V_m = -\int H \cdot dl$$

Magnetic vector potential

$$A = \frac{\mu}{4\pi}\iiint \frac{J}{r}\,dv$$

Lorentz force $F = Q\,(E + v \times B)$

Force on a differential current element

$$dF = I\,dl \times B$$

EXERCISE

1. State and explain Biot-Savart's law.
2. Obtain an expression for the magnetic field intensity due to infinitely long current carrying conductor.
3. Obtain an expression for the magnetic field intensity due to finite length current carrying conductor.

4. Obtain the expression for force between two current carrying conductors.
5. Derive an expression for magnetic field intensity at any point on the axis of circular coil carrying current.
6. Derive an expression for magnetic field intensity at any point on the axis of solenoid.
7. Derive the expression for magnetic field intensity on the axis of solenoid at (*a*) centre and (*b*) end point of the solenoid.
8. Derive an expression for magnetic field at the centre of toroidal coil.
9. Obtain the expression for torque in terms of magnetic moment.
10. Distinguish between scalar and vector magnetic potential.
11. Explain the vector magnetic potential and derive its expression.
12. A wire carrying a current of 100 A is bent into the form of a circle of diameter 10 cm. Calculate (*a*) the flux density at the centre of the coil and (*b*) the flux density at a point on the axis of the coil and 12 cm from it.
13. State and explain Ampere's circuital law.
14. A round copper conductor is carrying a current of 250 A. Determine the magnetising force and flux density at a distance of 10 cm from the conductor.
15. Show that the field strength at the end of a long solenoid is one half of that the centre.
16. A solenoid with radius of 2 cm is wound with 20 turns per cm and carries 10 mA. (*i*) find H at the centre of the solenoid if its length is 10 cm. (*ii*) if all the turns of the solenoid were compressed to make a ring of radius of 2 cm, what would be H at the centre of a ring. [*Ans:* 20 AT/m, 0.25 AT/m]
17. A solenoid of radius 10 cm and length 20 cm is wound uniformly with 200 turns of wire and carries a current of 5 A. Calculate the flux density B at the point on the axis at a distance 10 cm from the middle of the coil.
    [*Ans.* 2.23 kWb/m^2]
18. Calculate the magnetic field intensity at the centre of a square loop of side 1 m and carrying 20 A. Deduce the expression used. [*Ans.* H = 18 A/m]

19. A solenoid of 1000 turns is 40 cm long and 6 cm in diameter. If the coil carries a current of 100 mA, find the flux density (*a*) at the centre of the coil and (*b*) on the axis at one end of the coil.

[*Ans.* $B = 3.14 \times 10^{-4}$ Tesla, $B = 1.55 \times 10^{-4}$ Tesla]

20. Derive an expression for the inductance of solenoid and toroid.
21. Derive an expression for the inductance per meter length of two transmission lines.
22. Define self-inductance and mutual inductance. Show that $M = K\sqrt{L_1 L_2}$.
23. Derive an expression for inductance of coaxial cable.
24. State and explain Faraday's law.
25. Show that the torque on the planer current loop in a uniform magnitude field of flux density B is $\bar{m} \times \bar{B}$ where $\bar{m}$ is the magnitude dipole moment.
26. Obtain the expression for energy stored in magnetic field and also derive an expression for magnetic energy density.
27. An air coaxial transmission line has a solid inner conductor of radius '*a*' and very thin outer conductor of inner radius '*b*'. Determine the inductance per unit length of the line.
28. Derive the expressions for effective inductance of two inductors in series and parallel circuit when they are coupled.
29. A 4000 turns solenoid is 2 m long and has a diameter 10 cm. Calculate the inductance of the solenoid and energy stored when a current of 8A is flowing through the coil.
30. An iron ring of relative permittivity 100 is wound uniformly with two coils of 100 and 400 turns of wire. The cross-section of the ring is 4 cm and the mean length is 50 cm. Calculate the self-inductance of each of the two coils and the mutual inductance.
31. A 10 km transmission line consists of 1 cm diameter conductors spaced 1 m apart. Find the loop inductance of the system from first principles.
32. A square coil of area of 100 cm^2 and 200 turns is rotated about an axis at right angles to a uniform magnetic field of 0.7 Wb/m^2 at a uniform speed of 1000 rpm. Evaluate the expression for the instantaneous value of induced emf.

[*Ans.* 146.6 V]

33. An air solenoid 30 cm long and 1.5 m radius is wound on 1000 turns of closely spaced insulated wire over a length of 3 cm located near its centre (where the magnetic flux density is essentially constant). A separate coil of 100 turns of essentially the same radius is wound. What is the mutual inductance between two coils.

[*Ans.* 0.936 mH]

34. A 400 turn solenoid winding is 1 m long and has a diameter of 0.1 m. Calculate the self-inductance of the winding.

[*Ans.* 1.58 mH]

35. Two coils of self-inductances of 0.5 H and 0.8 H with negligible resistances are connected in series. If their mutual inductance is 0.2 H, determine the effective inductance of the combination.

[*Ans.* 1.7 H, 0.9 H]

36. Two identical coupled coils in series has an equivalent inductance values of 0.084 H and 0.0354 H. Find the values of self-inductances, mutual inductance and coupling coefficient.

[*Ans.* 29.85 mH, 29.85 mH, 12.15 mH, 0.41]

□□

4

ELECTRODYNAMIC FIELDS

Professor Oersted demonstrated that magnetic field is produced by changing electric current in 1820. Michael Faraday showed that an electric current is induced by changing the magnetic field in 1831. This phenomena is known as Electromagnetic Induction.

4.1. FARADAY'S LAW

The total electromagnetic force (emf) induced in a circuit is equal to the rate of decrease of the total magnetic flux linking the circuit.

$$v = -\frac{d\phi}{dt}$$

If 'N' is number of turns in the circuit, then the total flux is Nϕ, where ϕ is the magnetic flux.

$$v = -N\frac{d\phi}{dt}$$

By definition, the emf in a circuit is the line integral of the electric field around a closed path.

$$v = \oint E \cdot dl \quad \text{where E is the electric field.}$$

According to Gauss's law the total flux passing through the surface is equal to the surface integral of magnetic flux density over the surface.

$$\phi = \iint B \cdot ds$$

Substituting the value of ϕ in emf equation,

$$v = -\frac{d}{dt}\iint B \cdot ds$$

$$v = -\iint \frac{\partial B}{\partial t} \cdot ds$$

$$\text{But } v = \oint E \cdot dl$$

$$\oint E \cdot dl = -\iint_s \frac{\partial B}{\partial t} \cdot ds$$

Applying Stoke's theorem,

$$\oint E \cdot dl = \iint_s \nabla \times E \cdot ds$$

Then,

$$\iint_s \nabla \times E \cdot ds = -\iint_s \frac{\partial B}{\partial t} \cdot ds$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This is referred to as Maxwell's equation.

Faraday's law states that the total electromagnetic force induced in a circuit is equal to the rate of decrease of the total magnetic flux linking the circuit.

$$v = -\frac{d\phi}{dt}$$

If N is the number of turns in the circuit, the total flux is N ϕ. Then

$$v = -N\frac{d\phi}{dt}$$

An emf can be generated in a closed conducting loop by three ways.

(*i*) A stationary loop in a time varying magnetic field.
(*ii*) A moving loop in a static magnetic field.
(*iii*) A moving loop in a time varying magnetic field.

(i) Stationary loop in a time varying magnetic field:

When the conducting loop is stationary in a time varying magnetic field 'B', the emf is induced in the loop. The induced emf is called transformer emf.

$$v = N\int_s \frac{\partial B}{\partial t}\, ds$$

where N is the number of turns in a loop.

$$\text{If } N = 1$$

$$v = -\int_s \frac{\partial B}{\partial t}\, ds$$

It may be related to electric field E.

$$v = \oint E \cdot dl$$

Equating these two equations,

$$\oint E \cdot dl = -\int_s \frac{\partial B}{\partial t} \cdot ds$$

This is the integral form of Faradays law.

By applying Stoke's theorem,

$$\oint E \cdot dl = \int_s (\nabla \times E) \cdot ds$$

$$\therefore \quad \int_s (\nabla \times E)\, ds = -\int_s \frac{\partial B}{\partial t} \cdot ds$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This is the differential form of Faradays law. It states that a time varying magnetic field induces an electric field E whose curl is equal to the negative of the time derivation of B.

(ii) Moving loop in a static magnetic field:

Consider a conducting loop of length l moving across a static magnetic field B with a velocity v. The magnetic force F acting on any charged particle Q moving with a velocity v in a magnetic field B is given by

$$F = Q(v \times B)$$

The magnetic force is equivalent to the electrical force that would be exerted on the particle by an electric field E given by

$$E = \frac{F}{Q} = v \times B$$

The electric field E generated by the motion of the charged particle is called a motional electric field and it is perpendicular to v and B.

The induced voltage is induced emf and is given by

$$\int \mathrm{E} \cdot dl = \int (v \times \mathrm{B})\, dl$$

Example: Electric generator and motor.

Permanent magnet is used to produce a static magnetic field B. In the case of electric generator, emf is induced when the loop rotates in the magnetic field. It converts mechanical energy into electrical energy. In the case of electric motor, when a current is passing through the loop, loop starts to rotate in magnetic field. It converts electrical energy into mechanical energy.

(iii) Moving loop in a time varying magnetic field:

When the conducting loop is moving in a time varying magnetic field B, the induced emf is the sum of transformer emf and motional emf

$$\text{Total emf} = \oint \mathrm{E} \cdot dl = -\int_s \frac{\partial \mathrm{B}}{\partial t} \cdot ds + \oint_l (v \times \mathrm{B})\, dl$$

4.1.1. Transformer EMF

In a transformer, source of alternating current is applied to the primary winding. Due to this, the current in the primary winding (called as magnetizing current) produces alternating flux in the core of transformer. This alternating flux gets linked with the secondary winding, and because of the phenomenon of mutual induction an emf gets induced in the secondary winding. Magnitude of this induced emf can be found by using the following EMF equation of the transformer.

EMF equation of the Transformer

Let N_1 = Number of turns in primary winding

N_2 = Number of turns in secondary winding

ϕ_m = Maximum flux in the core (in wb) = $(B_m \times A)$

f = Frequency of the AC supply (in Hz)

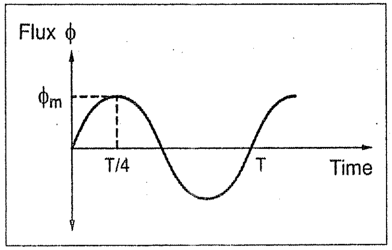


Fig. 4.1. Flux variation with time

As shown in Fig.4.1, the flux rises sinusoidally to its maximum value ϕ_m from 0, it reaches to the maximum value in one quarter of the cycle *i.e.,* in T/4 sec (where, T is time period of the sin wave of the supply = 1/*f*).

Therefore,

$$\text{Average rate of change of flux} = \frac{\phi_m}{T/4} = \frac{\phi_m}{1/4f}$$

Therefore,

$$\text{Average rate of change of flux} = 4f\,\phi_m \quad \text{(wb/s)}$$

Now, Induced emf per turn = Rate of change of flux per turn

Therefore,

$$\text{Average emf per turn} = 4f\,\phi_m \quad \text{(volts)}$$

Now, we know,

$$\text{Form factor} = \frac{\text{RMS value}}{\text{Average value}}$$

Therefore,

$$\text{RMS value of emf per turn} = \text{Form factor} \times \begin{Bmatrix}\text{Average emf} \\ \text{per turn}\end{Bmatrix}$$

As the flux ϕ varies sinusoidally, form factor of a sine wave is 1.11.

Therefore,

$$\text{RMS value of emf per turn} = 1.11 \times 4f\,\phi_m = 4.44f\,\phi_m$$

RMS value of induced emf in whole primary winding (E_1)

$$= \begin{Bmatrix}\text{RMS value of} \\ \text{emf per turn}\end{Bmatrix} \times \begin{Bmatrix}\text{Number of turns} \\ \text{in primary winding}\end{Bmatrix}$$

$$E_1 = 4.44f\,N_1\,\phi_m \quad \ldots (1)$$

Similarly, RMS induced emf in secondary winding (E_2) can be given as

$$E_2 = 4.44f\,N_2\,\phi_m \quad \ldots (2)$$

From the above equations (1) and (2),

$$\frac{E_1}{N_1} = \frac{E_2}{N_2} = 4.44f\,N_1\,\phi_m$$

This is called the ***emf equation of transformer,*** which shows, emf/number of turns is same for both primary and secondary winding.

For an ideal transformer on no-load, $E_1 = V_1$ and $E_2 = V_2$

where, V_1 = Supply voltage of primary winding

V_2 = Terminal voltage of secondary winding

Voltage Transformation Ratio (K)

As derived above, $\frac{E_1}{N_1} = \frac{E_2}{N_2} = K$

where, K = constant

This constant K is known as voltage transformation ratio.

- If $N_2 > N_1$, *i.e.,* $K > 1$, then the transformer is called step-up transformer.
- If $N_2 < N_1$, *i.e.,* $K < 1$, then the transformer is called step-up transformer.

4.1.2. Motional EMF

When a conductor is moved across a magnetic field, a potential difference is stepup across its ends. This potential difference is called ***'motional EMF'***.

Consider a wire of length 'l' moving across the magnetic field of induction B with a velocity V as shown in Fig.4.2.

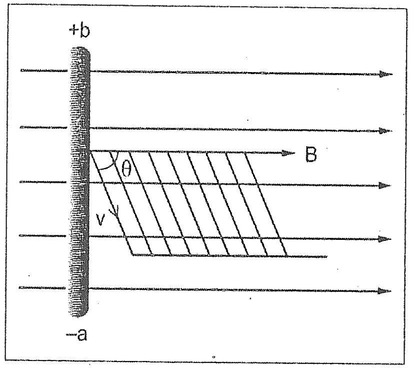


Fig. 4.2.

Force on Free Electron

Each free electron of the wire is moving within the wire and experience a force exerted by magnetic field.

$$\bar{F} = q\,(\bar{B} \times \bar{v})$$

These electrons gradually accumulate at the end "*a*" and leaving the other end "*b*". In this way point "*b*" acquires positive charge and point "*a*" acquires equal negative charge. This accumulation of electrons will continue till the force of electric field balances the force due to the motion of wire. Thus a potential difference is setup form point *b* and *a*.

We know that

Potential difference = Work done per unit charge

Let the total charge flows through the wire is "*q*" therefore

$$\text{Potential difference} = \frac{\text{Work}}{\text{Charge}} = \text{Force} \times \frac{\text{Distance}}{\text{Charge}}$$

$$e = q\,(\bar{B} \times \bar{v})\,l\,/\,q = l\,(\bar{B} \times \bar{v})$$

or

$$e = B\,v\,l\,\sin\theta$$

This is called motional EMF.

Therefore, Motional EMF = $B\,v\,l\,\sin\theta$

If the wire is moving at right angle to the field, $\theta = 90°$

Motional EMF = $B\,v\,l$

4.1.3. Faraday's Disc Generator

Faraday's disc generator consists of a circular disc of radius a rotating at a constant angular velocity ω radians per second in a magnetic flux density B. Two brushes make contact with the disc as shown in Fig.4.3.

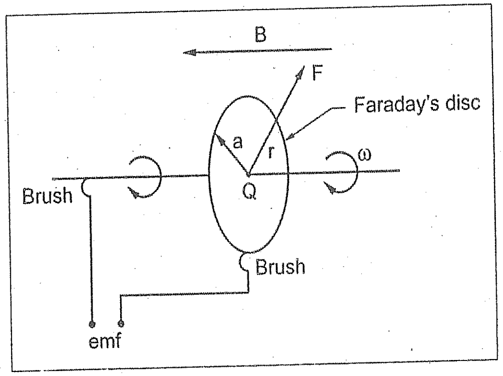


Fig. 4.3. Faraday's disc

Consider an electron of charge Q at any point in the disc at a radial distance r from the centre. As the disc rotates, the electron moves at a velocity $v = \omega\,r$ and experiences a force.

$$F = Q(v \times B)$$

The electric field acting upon the electron is

$$E(r) = \frac{F}{Q} = v \times B = \omega r B$$

The induced emf between the centre and the circumference of the disc is (*i.e.,* between two brushes).

$$\text{emf} = \int_0^a E(r)\,dr = \int_0^a \omega r B \cdot dr = \int_0^a B \omega r\,dr$$

$$\text{emf} = \frac{\omega B a^2}{2}$$

The mmf (number of ampere turns) required to produce a flux density B in a magnetic circuit is

$$\oint H\,dl = NI$$

4.1.4. Pull of an Electromagnet

Consider a horse-shoe type of electromagnet shown in Fig.4.4, and let the air-gap under each pole between the magnet and armature be l_g.

Let A be the area of cross-section of horse-shoe magnet. If the air-gap is increased by distance dl_g (armature away from the poles) the increased volume dv is $A\,dl_g$.

If F is the force of attraction between the magnet and the armature then the energy expended in displacing the armature through a distance dl_g is equal to $F \cdot dl_g$. (Force × Distance).

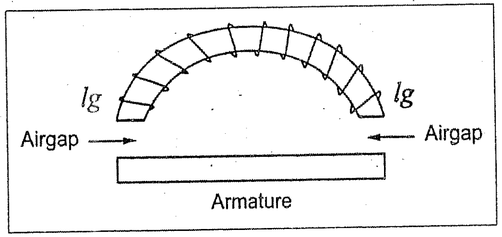


Fig. 4.4. Horse shoe magnet

$$\text{Energy stored per unit volume} = \frac{1}{2}\mu_0 H^2 = \frac{1}{2} BH = \frac{B^2}{2\mu_0} \text{ Joules/m}^3$$

$$\text{Increase in energy } d\omega = \left[\frac{B^2}{2\mu_0}\right] dv = \left[\frac{B^2}{2\mu_0}\right] \cdot (A \cdot dl_g)$$

But this is equal to $F \cdot dl_g$

$$F \cdot dl_g = \frac{B^2}{2\mu_0} \cdot (A\,dl_g)$$

The force required to pull the magnet

$$F = \frac{B^2 A}{2\mu_0} \text{ kgm f}$$

4.2. MAGNETIC CIRCUITS

An electric circuit forms a closed path or circuit through which the current flows. Magnetic flux are continuous and form closed paths. Hence a single flux line is a magnetic circuit.

The magnetic flux through magnetic circuit is defined as the ratio of magnetic motive force to the reluctance of the magnetic circuit.

$$\text{Magnetic flux} = \frac{\text{mmf}}{\text{Reluctance}}$$

Magnetic motive force of a magnetic circuit is equal to the line integral of magnetic field H around the closed circuit.

$$\text{mmf} = \oint H\,dl = NI \text{ Amp-turns}$$

The reluctance is defined as the ratio of total mmf of magnetic circuit to the flux through it.

$$\text{Reluctance} = \frac{\text{mmf}}{\text{Magnetic flux}}$$

$$\mathcal{R} = \frac{\oint H\,dl}{BA} = \frac{Hl}{BA} = \frac{Hl}{\mu H A} \qquad [\because B = \mu H]$$

$$\mathcal{R} = \frac{l}{\mu A} \text{ Henry}^{-1}$$

Reciprocal of reluctance is nothing but ***permeance.***

$$P = \frac{1}{\mathcal{R}} = \frac{\mu A}{l} \text{ Henry}$$

Magnetic Circuit with Air-gap

Consider an iron ring with a narrow gap of thickness t as shown in Fig.4.5.

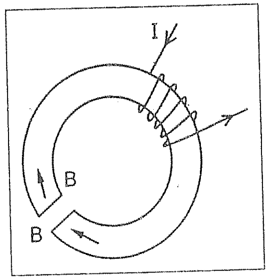


Fig. 4.5. Iron ring with air gap

By the continuity of the normal component of B the flux density in the gap is the same as in the iron. But the field intensity in the iron is

$$H_i = \frac{B}{\mu} = \frac{B}{\mu_0 \mu_r}$$

and the field intensity in the air-gap is,

$$H_g = \frac{B}{\mu_0}$$

4.3. ELECTRIC CURRENT

Electric current consists of two types of current namely, conduction current and displacement current.

4.3.1. Conduction Current

The conduction current I is flowing through a conductor whose resistance is R when potential V is applied across the conductor.

$$I_C = \frac{V}{R}$$

Resistance of the conductor is

$$R = \frac{\rho l}{A}$$

where, l is the length of the conductor
A is the area of cross-section of the conductor
ρ is the resistivity of the material

$$R = \frac{l}{\sigma A}$$

where, σ is conductivity of the material.

$$\sigma = \frac{1}{\rho}$$

If E is the electric field exist in the conductor, the potential

$$V = E l$$

$$\text{Conduction current } I_C = \frac{V}{R} = \frac{E l}{l} \sigma A = E \sigma A$$

Conduction current density is given by

$$J_C = \frac{I_C}{A} = \sigma E$$

$$J_C = \sigma E$$

$$\text{or} \quad J = \sigma E$$

This equation is known as the ***point form of Ohm's law.*** It states that the field strength within a conductor is proportional to the current density.

$$J \propto E$$

$$J = \sigma E$$

Conductivity

Conductivity may be defined as the current density per unit electric field

$$\sigma = \frac{1}{\rho}$$

Conductivity is the reciprocal of resistivity and it is given by

$$\sigma = \frac{l}{R A}$$

4.3.2. Displacement Current

The displacement current I_D is flowing through a capacitor when ac voltage is applied across the capacitor.

$$I_D = \frac{dQ}{dt} = C \frac{dV}{dt} \qquad [\because Q = CV]$$

For the parallel plate capacitor C

$$C = \frac{\varepsilon A}{d}$$

where, ε is permittivity of the medium

A is the area of the plate

d is the distance between two plates

$$\text{Displacement current } I_D = \frac{\varepsilon A}{d} \cdot \frac{dV}{dt}$$

$$= \varepsilon A \cdot \frac{dE}{dt} \qquad \left[\because \frac{V}{d} = E\right]$$

$$= A \cdot \frac{\partial D}{\partial t} \qquad [\because D = \varepsilon E]$$

$$\text{Displacement current density } J_D = \frac{I_D}{A}$$

$$J_D = \frac{\partial D}{\partial t}$$

4.3.3. Continuity of Current

Current is defined as the rate of movement of charge passing a given reference point.

$$I = \frac{dQ}{dt}$$

The increment of current ΔI crossing an increment surface Δs is given by

$$\Delta I = J \cdot \Delta s$$

where, J is the current density A/m^2

Total current is obtained by integrating

$$I = \int_s J \cdot ds$$

Consider a region bounded by a closed surface. The current through the closed surface is

$$I = \oint_s J \cdot ds$$

and this outward flow of positive charge must be balanced by a decrease of positive charge within the closed surface. If the charge inside the closed surface is denoted by Q, then the rate of decrease is $-\frac{dQ}{dt}$ and the principle of conservation of charges requires

$$I = \oint_s J \cdot ds = -\frac{dQ}{dt}$$

$$\oint_s J \cdot ds = -\frac{dQ}{dt}$$

This is the ***integral form of continuity equation.***

The differential or point form is obtained by changing the surface integral to a volume integral by using divergence theorem.

$$\oint_s J \cdot ds = \int_v (\nabla \cdot J)\, dv$$

Then,

$$\int_v (\nabla \cdot J)\, dv = -\frac{dQ}{dt}$$

$$= -\frac{d}{dt} \int_v \rho_v \, dv \qquad \left[\because Q = \int_v \rho_v \, dv\right]$$

$$\int_v (\nabla \cdot J)\, dv = -\int \frac{\partial \rho_v}{\partial t} \cdot dv$$

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$$

This is the ***point or differential form of continuity equation.***

4.4. MAXWELL'S EQUATIONS

Michael Faraday showed that the electric field is produced by a changing magnetic field. James Clerk Maxwell introduced a concept that the magnetic field is produced by a changing electric field. Maxwell derived four equations to describe the electromagnetic field. These four electromagnetic equations are also known as Maxwell's equations. These equations based on the fundamental laws of Gauss, Faraday and Ampere. Each of differential equations has its integral counterpart; one form may be derived from the other by using either Stoke's theorem or Divergence theorem. Maxwell's equations are derived as follows.

4.4.1. Maxwell's Equation from Ampere's Law

Ampere's circuital law: Ampere's law states that the line integral of magnetic field intensity H on any closed path is equal to the current enclosed by that path.

$$\oint_l H \cdot dl = I = \oiint_s J \, ds$$

In general the total current involves both conduction current and displacement current. A current through a resistive element is called conduction current whereas a current through a capacitive element is called displacement current.

Conduction current density: According to Ohm's law, the current through a conductor of resistance R is given by

$$I_C = \frac{V}{R}$$

$$\text{But} \quad R = \frac{\rho l}{A} = \frac{l}{\sigma A}$$

where, ρ is resistivity

σ is conductivity $(\sigma = \frac{1}{\rho})$

l is the length of the conductor

A is the area of cross-section of conductor

$$I_C = \frac{V \sigma A}{l}$$

If E is the electric field, then voltage $V = E \cdot l$

Substituting the value of V

$$I_C = E \sigma A$$

$$\frac{I_C}{A} = \sigma E$$

Conduction current density $(J_C = \frac{I_C}{A})$ is

$$J_C = \sigma E$$

Displacement current density:

The current through a capacitor is

$$I_D = \frac{dQ}{dt}$$

$$\text{But} \quad Q = CV$$

$$I_D = C \frac{dV}{dt}$$

The expression for capacitor is

$$C = \frac{\varepsilon A}{d}$$

where, ε is permittivity of the medium

A is area of the parallel plate of capacitor

d is distance between two plates

Substituting the value of C in the above equation

$$I_D = \frac{\varepsilon A}{d} \cdot \frac{dV}{dt}$$

$$\text{But} \quad V = E d$$

$$I_D = \frac{\varepsilon A}{d} \cdot d \cdot \frac{dE}{dt}$$

$$I_D = \varepsilon A \cdot \frac{dE}{dt}$$

$$\frac{I_D}{A} = \varepsilon \frac{\partial E}{\partial t} = \frac{\partial D}{\partial t} \qquad [\because D = \varepsilon E]$$

Displacement current density $(J_D = \frac{I}{A})$ is

$$J_D = \frac{\partial D}{\partial t}$$

Ampere's law can be written as

$$\oint H \cdot dl = \iint_S (J_C + J_D) \, ds$$

$$\text{then} \quad \oint H \cdot dl = \iint_S \left(\sigma E + \frac{\partial D}{\partial t} \right) ds$$

$$\oint H \cdot dl = \iint_S \left(\sigma E + \varepsilon \frac{\partial E}{\partial t} \right) ds$$

Unless or otherwise it is not specified J stands for conduction current density alone.

$$\text{Then,} \quad \oint H \cdot dl = \iint \left(J + \frac{\partial D}{\partial t} \right) ds$$

This is Maxwell's equation in integral form from Ampere's law. By applying Stoke's theorem,

$$\oint H \cdot dl = \iint_S \nabla \times H \cdot ds$$

Comparing the above two equations, then

$$\iint \nabla \times H \cdot ds = \iint \left(J + \frac{\partial D}{\partial t} \right) ds$$

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

This is Maxwell's equation in differential or point form from Ampere's law.

Statement:* *The magnetomotive force around a closed path is equal to the sum of the conduction current and displacement current enclosed by the path.

In other words, ***magnetic voltage around a closed path is equal to the electric current through the path.***

4.4.2. Maxwell's Equation From Faraday's Law

Faraday's Law: Faraday's law states that the electromagnetic force (mmf) induced in a circuit is equal to the rate of decrease of the magnetic flux linkage the circuit.

$$\text{i.e.,} \quad v = -\frac{d\phi}{dt} = \frac{d}{dt}\left(\iint_S B \cdot ds\right)$$

$$\text{But} \quad v = \oint E \cdot dl$$

$$\oint E \cdot dl = -\frac{d}{dt} \iint_S B \cdot ds$$

$$\oint E \cdot dl = -\iint \frac{\partial B}{\partial t} \cdot ds \qquad [\because B = \mu H]$$

$$\text{or} \quad \oint E \cdot dl = -\mu \iint \frac{\partial H}{\partial t} \cdot ds$$

This is Maxwell equation in integral form. By applying Stoke's theorem,

$$\oint E \cdot dl = \iint_S \nabla \times E \cdot ds$$

Comparing both the equations,

$$\iint_S \nabla \times E \cdot ds = -\iint \frac{\partial B}{\partial t} \cdot ds$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

This is the Maxwell's equation from Faraday's law in differential or point form.

Statement:* *The electromotive force around a closed path is equal to the magnetic displacement (flux density) through that closed path.

In other words, ***electric voltage around a closed path is equal to the magnetic current through the path.***

4.4.3. Maxwell's Equation from Electric Gauss's Law

Electric Gauss's law: Gauss's law states that electric flux emerging through any closed surface is equal to the charge enclosed by the surface.

$$\chi = Q$$

$$\iint D \cdot ds = Q$$

$$\text{or} \quad \iiint \rho_v \cdot dv = Q$$

$$\text{Then} \quad \iint_S D \cdot ds = \iiint_v \rho_v \cdot dv$$

This is the Maxwell's equation from electric Gauss's law in integral form.

By applying Divergence theorem,

$$\iint_S D \cdot ds = \iiint_v \nabla \cdot D\, dv$$

Comparing the above two equations,

$$\iiint_v \nabla \cdot D\, dv = \iiint_v \rho \cdot dv$$

$$\nabla \cdot D = \rho$$

This is the Maxwell's equation from electric Gauss's law in differential or point form.

Statement: The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

4.4.4. Maxwell's Equation from Magnetic Gauss's Law

Magnetic Gauss's law: Magnetic Gauss's law states that the total magnetic flux through any closed surface is equal to zero.

$$\phi = 0$$

$$\iint_S B \cdot ds = 0$$

This is the Maxwell's equation in integral form from magnetic Gauss's law.

By applying Divergence theorem,

$$\iint_S B \cdot ds = \iiint_v \nabla \cdot B\, dv$$

Comparing the above two equations

$$\iiint_v \nabla \cdot B = 0$$

$$\nabla \cdot B = 0$$

This is the Maxwell's equation in differential or point form from magnetic Gauss's law.

Statement: The net magnetic flux emerging through any closed surface is zero.

Maxwell's Equations

Maxwell's equations are summarised as follows:

	Differential form	Integral form
I.	$\nabla \times H = J + \frac{\partial D}{\partial t}$ or $\nabla \times H = \sigma E + \varepsilon \frac{\partial \varepsilon}{\partial t}$	$\oint H \cdot dl = \iint_S \left(J + \frac{\partial D}{\partial t} \right) ds$ $\oint H \cdot dl = \iint_S \left(\sigma E + \varepsilon \frac{\partial E}{\partial t} \right) ds$
II.	$\nabla \times E = - \frac{\partial B}{\partial t}$ or $\nabla \times E = - \mu \frac{\partial H}{\partial t}$	$\oint E \cdot dl = \iint_S \frac{\partial B}{\partial t} \cdot ds$ $\oint E \cdot dl = - \mu \iint_S \frac{\partial H}{\partial t} \cdot ds$
III.	$\nabla \cdot D = \rho$	$\oiint_s D \cdot ds = \iiint_v \rho\, dv$
IV.	$\nabla \cdot B = 0$	$\oiint_s B \cdot ds = 0$

In a free-space, there is no charges enclosed and since it is a dielectric, there is no conductivity in the medium. *i.e.,* $\rho = 0$ and $\sigma = 0$ conduction current is zero. Then, Maxwell's equation becomes in free space as follows:

	Differential form	Integral form
I.	$\nabla \times H = \frac{\partial D}{\partial t}$	$\oint H \cdot dl = \iint_S \frac{\partial D}{\partial t} \cdot ds$
II.	$\nabla \times E = - \frac{\partial B}{\partial t}$	$\oint E \cdot dl = - \iint_S \frac{\partial B}{\partial t} \cdot ds$
III.	$\nabla \cdot D = 0$	$\oiint_s D \cdot ds = 0$
IV.	$\nabla \cdot B = 0$	$\oiint_s B \cdot ds = 0$

4.4.5. Maxwell's Equations in Phasor Form

Time varying field: The electric field and magnetic field may be assumed as sinusoidal time varying quantities. The time varying field may be expressed in terms of phasor quantity.

$$E(x,t) = \text{Real part of } [E(x)\, e^{j\omega t}]$$

$$\frac{\partial E}{\partial t}(x,t) = \text{Real part of } [j\omega E(x)\, e^{j\omega t}]$$

Apply for Maxwell's equation,

$$\text{Real part of } [\nabla \times H] = \text{Real part of } [(\sigma E + j\omega\varepsilon E)\, e^{j\omega t}]$$

$$\nabla \times H = \sigma E + j\omega\varepsilon E$$

$$\nabla \times H = (\sigma + j\omega\varepsilon) E$$

For magnetic field,

$$H(x,t) = \text{Re}\, [H(x)\, e^{j\omega t}]$$

$$\frac{\partial H}{\partial t}(x,t) = \text{Re}\, [j\omega H(x)\, e^{j\omega t}]$$

Apply for Maxwell's equation

$$\text{Re}\, [\nabla \times E] = -\text{Re}\, [j\omega\mu\, H\, e^{j\omega t}]$$

$$\nabla \times E = -j\omega\mu H$$

Maxwell's equations in phasor form are as follows:

Differential Form	Integral Form
$\nabla \times H = (\sigma + j\omega\varepsilon) E$	$\oint H \cdot dl = \iint (\sigma + j\omega\varepsilon E)\, ds$
$\nabla \times E = -j\omega\mu H$	$\oint E \cdot dl = -\mu \iint j\omega H \cdot ds$
$\nabla \cdot D = \rho$	$\oint D \cdot ds = \iiint \rho\, dv$
$\nabla \cdot B = 0$	$\oint B \cdot ds = 0$

4.5. BOUNDARY CONDITIONS

Conditions at a boundary surface

The following statements can be made regarding the electric and magnetic fields at any surface of discontinuity.

1. The tangential component of electric field E is continuous at the surface *i.e.*, it is the same just outside the surface as it is just inside the surface.
2. The tangential component of magnetic field H is continuous across a surface except at the surface of perfect conductor. At the surface of a perfect conductor the tangential component H is discontinuous by an amount equal to the linear current density.
3. The normal component of electric flux density D is continuous if there is no surface charge density. Otherwise D is discontinuous by an amount equal to the surface charge density.
4. The normal component of magnetic flux density B is continuous at the surface of discontinuity.

The proof of these boundary conditions is obtained by the application of Maxwell's equations at the boundary between the media. Consider a small rectangle of width Δx and length Δy at the boundary enclosing a small portion of each of media as shown in Fig.4.6.

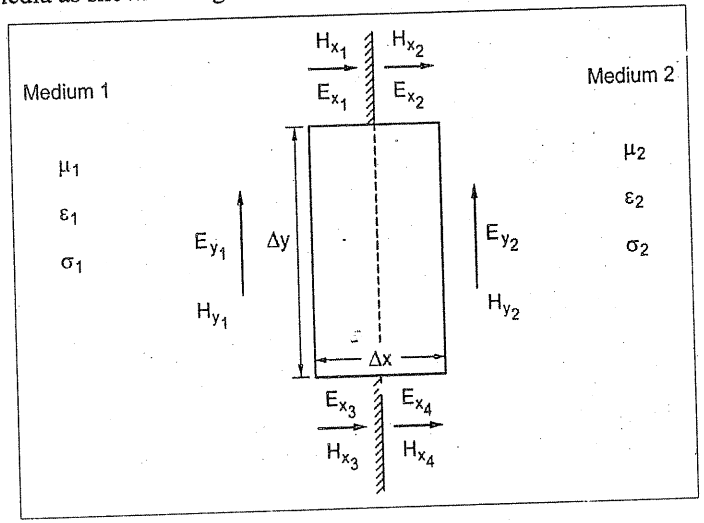


Fig. 4.6. A boundary surface between two media

The integral form of the second Maxwell's equation is

$$\oint E \cdot dl = \iint_S \frac{\partial B}{\partial t} \cdot ds$$

This is applied to a rectangle

$$E_{y_1}\,\Delta y + E_{x_1}\frac{\Delta x}{2} + E_{x_2}\frac{\Delta x}{2} - E_{y_2}\,\Delta y - E_{x_4}\frac{\Delta x}{2} - E_{x_3}\frac{\Delta x}{2} = \frac{\partial B}{\partial t}\,\Delta x\,\Delta y$$

Consider the area of the rectangle is made to approach zero by reducing the width Δx to approach zero.

Then, $$E_{y_1}\,\Delta y - E_{y_2}\,\Delta y = 0$$

$$E_{y_1} = E_{y_2}$$

The tangential component of E is continuous.

The integral form of first Maxwell's equation is

$$\oint_l H \cdot dl = \iint_S \left(J + \frac{\partial D}{\partial t}\right) ds$$

Applying to the rectangle,

$$H_{y_1}\,\Delta y + H_{x_1}\frac{\Delta x}{2} + H_{x_2}\frac{\Delta x}{2} - H_{y_2}\,\Delta y - H_{x_4}\frac{\Delta x}{2} - H_{x_3}\frac{\Delta x}{2} = \left(J + \frac{\partial D}{\partial t}\right)\Delta x\,\Delta y$$

If $\Delta x \to 0$, then $$H_{y_1}\,\Delta y - H_{y_2}\,\Delta y = 0$$

$$H_{y_1} = H_{y_2}$$

The tangential component of H is continuous.

For a perfect conductor, a high frequency current will flow in a thin sheet near the surface. In a current sheet, a linear current density J_l flows in a sheet of depth Δx. It is given in terms of surface current density J A/m^2.

$$\underset{\Delta x \to 0}{\mathrm{Lt}}\; J \cdot \Delta x = J_l \quad \text{A/m}$$

If the Maxwell's I equation is applied to the rectangle, then

$$H_{y_1}\,\Delta y + H_{x_1}\frac{\Delta x}{2} + H_{x_2}\frac{\Delta x}{2} - H_{y_2}\,\Delta y - H_{y_4}\frac{\Delta x}{2} - H_{y_3}\frac{\Delta x}{2} = \left(J + \frac{\partial D}{\partial x}\right)\Delta x\,\Delta y$$

$$= J\,\Delta x\,\Delta y + \frac{\partial D}{\partial x}\,\Delta x\,\Delta y$$

$$= J_l\,\Delta y + \frac{\partial D}{\partial x}\,\Delta x\,\Delta y$$

As Δx approaches 0, (i.e., $\Delta x \to 0$)

$$H_{y_1}\,\Delta y - H_{y_2}\,\Delta y = J_l\,\Delta y$$

$$H_{y_1} - H_{y_2} = J_l$$

The tangential component of H is discontinuous by an amount of linear current density (A/m) at the surface of a perfect conductor.

Consider a pill box of volume $ds \cdot \Delta x$ at the boundary between the two media as shown in Fig.4.7, where ds is the area of flat surfaces of pill box and Δx is their separation. ρ is the volume charge density enclosed by the volume.

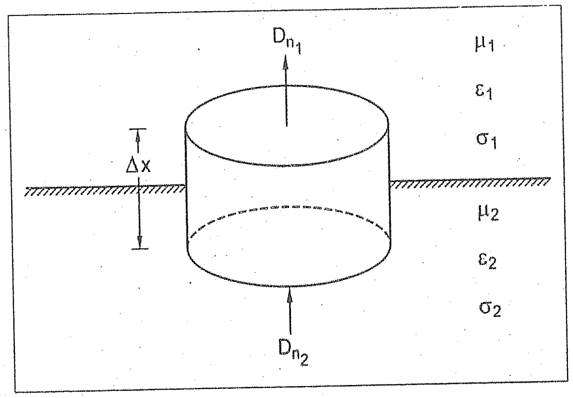


Fig. 4.7. A pill box at the boundary surface

The integral form of the Maxwell's 3rd equation

$$\oiint_s D \cdot ds = \iiint_v \rho\, dv$$

Apply to the pill box at the boundary

$$D_{n_1}\,ds - D_{n_2}\,ds = \rho\, ds \cdot \Delta x$$

As $\Delta x \to 0$ *i.e.*, the flat surfaces of the box are squeezed together

$$D_{n_1}\,ds - D_{n_2}\,ds = 0$$

$$D_{n_1} = D_{n_2}$$

The normal component of D is continuous if there is no surface charge density.

In the case of perfect conductor, the surface has the surface charge density ρ_s coulomb/m^2. The volume charge density ρ_v can be written as, if $\Delta x \to 0$.

$$\underset{\Delta x \to 0}{\text{Lt}} \ \rho_v \cdot \Delta x = \rho_s$$

If the Maxwell's third equation is applied to the pill box

$$D_{n_1} ds - D_{n_2} ds = \rho_v \ ds \cdot \Delta x = \rho_s \cdot ds$$

As $\Delta x \to 0$,

$$D_{n_1} - D_{n_2} = \rho_s$$

The normal component of D is discontinuous across the boundary by the amount of surface charge density.

The integral form of Maxwell's fourth equation is

$$\iint B \cdot ds = 0$$

Apply to the Pill box at the boundary,

$$B_{n_1} ds - B_{n_2} ds = 0$$

$$B_{n_1} = B_{n_2}$$

The normal component of magnetic flux density B is always continuous across the boundary.

4.6. RELATION BETWEEN CIRCUIT THEORY AND FIELD THEORY

Electromagnetics is important because it provides on real world, three dimensional understanding of electricity and magnetism Electric and magnetic fields forming a bridge from circuit theory to the transmission lines, waves and wireless systems.

Circuit theory treats resistors, capacitors and inductors as two terminal devices connected by the wires. Field theory deals with the space both inside and outside these devices, providing a three dimensional, real world understanding of how they work. Most of the energy flows outside the conductors of a transmission line rather than inside them as usually postulated by circuit theory. If the spacing of the conductors of a two wire transmission line is increased sufficiently, most of the energy leaves the conductors and is radiated in to space, from a radiating system or antenna.

The comparison between circuit theory and field theory is given as follows.

Comparison between Circuit theory and Field theory

S.No.	Circuit theory	Field theory
1.	This analysis originated by its own.	Evolved from Transmission theory.
2.	Applicable only for portion of RF range.	Beyond RF range (microwave).
3.	The dependent and independent parameters, I and V are directly obtained for the given circuit.	Not directly, through E and H.
4.	Parameters of medium are not involved.	Parameter of medium (permittivity and permeability) are involved in the analysis.
5.	Laplace Transform is employed.	Maxwell Equation is employed.
6.	Z, Y and H parameters are used.	S parameter is used.
7.	Low power is involved.	Relatively high power is involved.
8.	Simple to understand.	Needs visualization ability.
9.	Two dimensional analysis.	Three-dimensional analysis.
10.	Frequency is used as reference.	Wavelength is used as reference.
11.	Lumped components are involved.	Distributed components are involved.

SOLVED EXAMPLES

Example 4.1 *For 1 A conductor current in copper wire, find the corresponding displacement current at 100 MHz. Assume for copper $\sigma = 5.8 \times 10^7$ mho/m.*

☺ ***Solution:*** Conduction current $I_C = J_C A = 1$ Amp

$$J_C = \frac{I}{A} = \sigma E$$

$$E = \frac{J_C}{\sigma} = \frac{I/A}{\sigma} = \frac{0.172 \times 10^{-7}}{A} \text{ V/m}$$

Displacement current $I_D = \omega \varepsilon E \cdot A = \omega \varepsilon_0 \varepsilon_r E A$

For copper $\varepsilon_r = 1$, $I_D = 2\pi \times 100 \times 10^6 \times \frac{10^{-9}}{36\pi} \times \frac{0.172 \times 10^{-7}}{A} A$

$$\mathbf{I_D = 9.556 \times 10^{-11}\ A}$$ **Ans.** ☜ ☺☺

Example 4.2 *Write the Maxwell's equations in integral, point and phasor form and also indicate the associated laws.*

☺ ***Solution:***

Maxwell's equations: ***Point form:***

$$\nabla \times H = J + \frac{\partial D}{\partial t} = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \cdot D = \rho_v$$

$$\nabla \cdot B = 0$$

Integral form:

$$\oint H \cdot dl = \iint \left(J + \frac{\partial D}{\partial t}\right) ds = \iint \left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right) ds$$

$$\oint E \cdot dl = -\iint \frac{\partial B}{\partial t} ds = -\mu \iint \frac{\partial H}{\partial t} ds$$

$$\oiint_s D \cdot ds = \iiint \rho_v \cdot dv$$

$$\oiint_s B \cdot ds = 0$$

Phasor form:

Point form	**Integral form**
$\nabla \times H = (\sigma + j\omega\varepsilon) E$	$\oint H \cdot dl = \iint (\sigma + j\omega\varepsilon) E \cdot ds$
$\nabla \times E = -j\omega\mu H$	$\oint E \cdot dl = -\mu \iint j\omega H \cdot ds$
$\nabla \cdot D = \rho_v$	$\oint D \cdot ds = \iiint \rho_v\, dv$
$\nabla \cdot B = 0$	$\oint B \cdot ds = 0$

Associated laws:

Maxwell's equation : Ampere's circuital law
II equation : Faraday's law
III equation : Electric Gauss's law
IV equation : Magnetic Gauss's law

Example 4.3 *Write down the boundary conditions at the interface between different media in both scalar and vector forms.*

☺ ***Solution:*** The following statements can be made regarding the electric and magnetic fields at any surface of discontinuity.

1. The tangential component of electric field E is continuous at the surface *i.e.,* it is the same just outside the surface as it is just inside the surface.
2. The tangential component of magnetic field H is continuous across a surface except at the surface of perfect conductor. At the surface of a perfect conductor, the tangential component H is discontinuous by an amount equal to the linear current density.
3. The normal component of electric flux density D is continuous if there is no surface charge density. Otherwise D is discontinuous by an amount equal to the surface charge density.
4. The normal component of magnetic flux density B is continuous at the surface of discontinuity.

$$E_{y_1} = E_{y_2}$$

$$H_{y_1} = H_{y_2}$$

$$H_{y_1} - H_{y_2} = J_l$$

$$D_{n_1} = D_{n_2}$$

$$D_{n_1} - D_{n_2} = \rho_s$$

$$B_{n_1} = B_{n_2}$$

(ii) *What are the salient points to be noted when the boundary conditions are applied?*

It should be noted that the flux does not pass through the conductor and the net charge on the conductor remains zero.

Under static condition all net charge will be on the outer surfaces of a conductor and both E and D are therefore zero within the conductor. ☺☺

Example 4.4 *In a material for which $\sigma = 5.0$ s/m and $\varepsilon_r = 1$ and $E = 250 \sin 10^{10} t$ (V/m). Find the conduction and displacement current densities.*

☺ ***Solution:***

$$\sigma = 5 \text{ s/m}, \quad \varepsilon_r = 1$$

$$E = 250 \sin 10^{10} t \text{ (V/m)}$$

Conduction current density $J_C = \sigma E$

$$= \mathbf{1250 \sin 10^{10} t \ (V/m)} \quad \textbf{Ans.}$$ ☜

Displacement current density $J_D = \dfrac{\partial D}{\partial t} = \varepsilon_0 \dfrac{\partial E}{\partial t}$

$$= \frac{10^{-9}}{36\pi} \times 10^{10} \times 250 \cos 10^{10} t \quad \text{(V/m)}$$

$$J_D = \mathbf{22.1 \cos 10^{10} t \quad V/m} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 4.5 *Find the total current in a circular conductor of radius 4 mm if the current density varies according to $J = \dfrac{10^4}{r}$ A/m^2.*

☺ ***Solution:***

$$J = \frac{10^4}{r}$$

Current $I = \int J \cdot ds$

$$= \int_{\phi=0}^{2\pi} \int_{r=0}^{0.004} \frac{10^4}{r} \cdot r \, dr \, d\phi = 10^4 \int_{\phi=0}^{2\pi} d\phi \int_{0}^{0.004} dr$$

$$= 10^{-4} \times 0.004 \int_{\phi=0}^{2\pi} d\phi$$

$$= 10^{-4} \times 0.004 \times [\phi]_0^{2\pi} = 10^{-4} \times 0.004 \times 2\pi$$

$$= \mathbf{80\pi} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 4.6 *The magnetic field intensity in free space is given as $H = H_0 \sin\theta \, \bar{a}_y$ A/m, where $\theta = \omega t - \beta z$ and β is a constant quantity. Determine the displacement current density.*

☺ ***Solution:***

$$H = H_0 \sin\theta \ \bar{a}_y \quad \text{A/m}$$

$$H = H_0 \sin(\omega t - \beta z)\,\bar{a}_y \quad \text{A/m}$$

The displacement current in free space is given by

$$\nabla \times H = J$$

$$\nabla \times H = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & H_0 \sin\theta & 0 \end{vmatrix}$$

$$\nabla \times H = \bar{a}_x \left[0 - \frac{\partial}{\partial z}(H_0 \sin\theta) \right] + 0 + \bar{a}_z \left[\frac{\partial}{\partial x}(H_0 \sin\theta) - 0 \right]$$

$$= -\bar{a}_x H_0 \frac{\partial}{\partial z}[\sin(\omega t - \beta z)] + \bar{a}_y H_0 \frac{\partial}{\partial x}[\sin(\omega t - \beta z)]$$

$$= -H_0 \beta \cos(\omega t - \beta z)\ \bar{a}_x$$

$$\mathbf{J = -\beta H_0 \cos(\omega t - \beta z)\,\bar{a}_x} \quad \textbf{Ans.}$$ ☜ ☺☺

Example 4.7 *A Conductor 1 cm in length is parallel to z-axis and rotates at radius of 25 cm at 1200 pm. Find induced voltage, if the radial field is given by $B = 0.5\, a_r$ T.*

☺ ***Solution:*** Let $E = v \times B$

In one minute, there are 1200 revolutions which corresponds to 20 revolution in one second. In one revolution, distance travelled is $2\pi r$ meters.

Hence in 20 revolutions, the distance travelled in one second is $40\pi r$ meters.

$$v = (40\pi r)\, a_\phi = 40\pi\,(25 \times 10^{-2})\, a_\phi$$

$$v = 31.416\, a_\phi \quad \text{m/s}$$

$$E = v \times B = 31.416\, a_\phi \times 0.5\, a_r$$

$$E = 15.708\,(-a_z)$$

The induced voltage is given by

$$e = \oint \mathrm{E} \cdot dl$$

$$dl = (dz)\, a_z$$

As the conductor is parallel to z axis

$$e = \int_{z=0}^{0.01} 15.708\,(-a_z)\,(dz)\,a_z = -15.708\,[z]_0^{0.01}$$

$$e = -\mathbf{157.08\ mV}$$ **Ans.** ☞

☺☺

Example 4.8 *Show that the ratio of the amplitudes of the conduction current density and displacement current density is σ/ωε. Find the current amplitude ratio if the applied field is $E = E_m e^{-t/\lambda}$ where λ is real.*

☺ ***Solution:*** The conduction current density is given by

$$J_C = \sigma\,E = \sigma\,E_m \cos\omega t$$

The displacement current density is given by

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial(\varepsilon E)}{\partial t} = \varepsilon\,\frac{\partial}{\partial t}\,[E_m \cos\omega t\,] = \omega\,\varepsilon\,E_m \sin\omega t$$

The ratio of the amplitude of two densities is given by

$$\left|\frac{J_C}{J_D}\right| = \frac{\sigma\,E_m}{\omega\,\varepsilon\,E_m} = \frac{\sigma}{\omega\,\varepsilon}$$

If the applied field $E = E_m\,e^{-t/\lambda}$

$$J_C = \sigma\,E = \sigma\,E_m\,e^{-t/\lambda}$$

$$J_D = \varepsilon\,\frac{\partial E}{\partial t}$$

$$= \varepsilon\,E_m\left[-\frac{1}{\lambda}\right]e^{-t/\lambda} = \frac{\varepsilon\,E_m}{\lambda}\,e^{-t/\lambda}$$

The ratio of amplitudes of two densities is given by

$$\frac{J_C}{J_D} = \frac{\sigma\,E_m}{\frac{\varepsilon\,E_m}{\lambda}} = \frac{\sigma\,\lambda}{\varepsilon}$$

☺☺

Example 4.9 *If the following are the components of the electric field vector of Maxwell's equations $E_y = A\,e^{j\omega(t-\frac{z}{v})}$ $E_x = 0$ and $E_z = 0$, what are the components of magnetic vector H?*

☺ ***Solution:***

$$E_y = A\,e^{j\omega(t-\frac{z}{v})}$$

$$E_x = E_z = 0$$

By Maxwell's equation, $\nabla \times E = -\dfrac{\partial B}{\partial t} = -\mu\,\dfrac{\partial H}{\partial t}$

$$\nabla \times E = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\bar{a}_z\,\frac{\partial E_y}{\partial z}$$

$$-\bar{a}_x\,\frac{\partial E_y}{\partial z} = -\mu\,\frac{\partial H_z}{\partial t}$$

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu}\,\bar{a}_x\,\frac{\partial}{\partial z}\left[A \cdot e^{j\omega(t-\frac{z}{v})}\,\right]$$

$$= -\frac{1}{\mu\,v}\,\bar{a}_x\,j\omega\,A\,e^{j\omega(t-\frac{z}{v})}$$

$$H_z = \bar{a}_x\,\frac{j\omega A}{\mu\,v}\int e^{j\omega(t-\frac{z}{v})}\,dt$$

$$= \bar{a}_x\,\frac{j\omega A}{\mu\,v}\,\frac{e^{j\omega(t-\frac{z}{v})}}{j\omega}$$

$$H_z = \bar{a}_x\,\frac{1}{\mu\,v}\,A\,e^{j\omega(t-\frac{z}{v})}$$

$$= \bar{a}_x\,\frac{\sqrt{\mu\,\varepsilon}}{\mu}\,A\,e^{j\omega(t-\frac{z}{v})} \qquad \left[\because v = \frac{1}{\sqrt{\mu\,\varepsilon}}\right]$$

$$= \bar{a}_x \sqrt{\frac{\varepsilon}{\mu}}\ A\, e^{j\omega\left(t-\frac{z}{v}\right)}$$

$$\mathbf{H}_z = \bar{a}_x \frac{\mathbf{A}}{z_0} e^{j\omega\left(t-\frac{z}{v}\right)} \quad \mathbf{A/m} \qquad \left[\because z_0 = \sqrt{\frac{\mu}{\varepsilon}}\right]$$

☺☺

Example 4.10 *A sinusoidal plane wave is transmitted through a medium whose electric field strength is 10 kV/m and relative permittivity of the medium is 4. Determine the mean rms power flow/unit area.*

☺ ***Solution:***

$$E = 10 \times 10^3 \ \text{V/m}$$

$$\varepsilon_r = 4, \quad \mu_r = 1$$

$$\frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0 \sqrt{\frac{1}{\varepsilon_r}} \qquad \left[\because \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\,\pi\right]$$

$$= \frac{120\,\pi}{\sqrt{4}} = 188.5 \text{ ohms}$$

$$\therefore \quad H = \frac{E}{188.5}$$

Power flow/unit area, $P = E \times H$

$$= E \times \frac{E}{188.5} = \frac{(10 \times 10^3)^2}{188.5}$$

$$= \mathbf{530.503 \ kW/m^2} \quad \textbf{Ans.}$$ ☞

☺☺

Example 4.11 *Find the displacement current density within a parallel plate capacitor where $\varepsilon = 10\,\varepsilon_0$, $A = 0.01\ m^2$ and $d = 5$ cm and the capacitor voltage is 100 sin 314 t volts.*

☺ ***Solution:*** Displacement current density

$$J_D = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t}$$

$$E = \frac{V}{d}$$

$$J_D = \frac{\varepsilon}{d} \cdot \frac{\partial V}{\partial t} = \frac{\varepsilon}{d} \frac{\partial}{\partial t} (100 \sin 314\, t)$$

$$= \frac{100\,\varepsilon}{d} \cos(314\,t) \cdot 314$$

$$= \frac{100}{5 \times 10^{-2}} \times \frac{10}{36\,\pi \times 10^9} \times 314 \cos(314\,t)$$

$$= 5.55 \times 10^{-5} \cos(314\,t) \quad \text{A/m}^2$$

Displacement current $I_D = J_D \cdot A$

$$= 5.55 \times 10^{-5} \times 0.01 \cos(314\,t) \text{ A}$$

$$= \mathbf{5.55 \times 10^{-7} \cos(314\,t) \ A} \quad \textbf{Ans.}$$ ☞ ☺☺

Example 4.12 *The conduction current flowing through a wire with conductivity $\sigma = 3 \times 10^7$ s/m and relative permeability $\varepsilon_r = 1$ is given by $I_C = 3 \sin \omega t$ (mA). If $\omega = 10^8$ rad/sec. Find the displacement current.*

☺ ***Solution:*** Conduction current is given by

$$I_C = \sigma E A$$

$$E = \frac{I_C}{\sigma A} = \frac{3 \times 10^{-3} \sin \omega t}{3 \times 10^7 \times A} = \frac{1 \times 10^{-10}}{A} \sin \omega t$$

$$\frac{\partial E}{\partial t} = \frac{1 \times 10^{-10}}{A} \omega \cos \omega t$$

$$J_D = \varepsilon \frac{\partial E}{\partial t}$$

$$J_D = \varepsilon\,\omega \cdot \frac{1 \times 10^{-10}}{A} \cos \omega t$$

$$= 8.85 \times 10^{-12} \times 10^8 \times \frac{10^{-16}}{A} \cos 10^8\, t$$

Displacement current $I_D = J_D \cdot A$

$$= 8.85 \times 10^{-12} \times 10^8 \times \frac{10^{-16}}{A} \cos 10^8\, t \ \text{A}$$

$$\mathbf{I_D = 8.85 \times 10^{-4} \cos 10^8\, t \ Amperes} \quad \textbf{Ans.}$$ ☞ ☺☺

Example 4.13 *A magnetic field $H = \begin{cases} 3 \cos x\, \bar{a}_x + z \cos x\, \bar{a}_y \ A/m, & \text{for } z \geq 0 \\ 0, & \text{for } z < 0 \end{cases}$ is applied to a perfectly conducting surface in xy plane. Find the current density on the conductor surface.*

☺ **Solution:** From Ampere's circuit law,

$$\text{Current density } J = \nabla \times H$$

$$\text{In cartesian form, } J = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\bar{a}_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\bar{a}_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\bar{a}_z$$

From H (given), $H_x = 3 \cos x$

$H_y = z \cos x$

$H_z = 0$

$$J = \left[0 - \frac{\partial z \cos x}{\partial z}\right]\bar{a}_x + \left[\frac{\partial\, 3 \cos x}{\partial z} - 0\right]\bar{a}_y + \left[\frac{\partial z \cos x}{\partial x} - \frac{\partial\, 3 \cos x}{\partial y}\right]\bar{a}_z$$

$$= -\cos x\, \bar{a}_x - z \sin x\, \bar{a}_z \quad \text{A/m}^2$$

$$\mathbf{J = -\cos x\, \bar{a}_x - z \sin x\, \bar{a}_z \quad A/m^2, \quad for\ z \geq 0}$$

$$\mathbf{= 0, \quad for\ z < 0}$$ **Ans.** ☜ ☺☺

Example 4.14 *If $D = 20\, x\, \bar{a}_x - 15\, y\, \bar{a}_y + k\, z\, \bar{a}_z\ \mu C/m^2$, find the value of k to satisfy the Maxwell's equations for region $\sigma = 0$ and $\rho_v = 0$.*

☺ **Solution:** As $\sigma = 0$ and $\rho_v = 0$, the medium in which D and B are present is nothing but free space. So the Maxwell's equations obtained from Gauss Law is given by

$$\nabla \cdot D = \rho_v = 0 \quad \text{for free space}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0$$

$$\frac{\partial}{\partial x}(20\, x) + \frac{\partial}{\partial y}(-15\, y) + \frac{\partial}{\partial z} k(z) = 0$$

$$20 - 15 + k = 0$$

$$\mathbf{k = -5\ \mu C/m^2}$$ **Ans.** ☜ ☺☺

Example 4.15 *If the magnetic field $H = (3\, x \cos\beta + 6\, y \sin\alpha)\, \bar{a}_z$, find the current density J if fields are invariant with time.*

☺ **Solution:** The point form of Maxwell's second equation is

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

If the fields are invariant with time $\frac{\partial D}{\partial t} = 0$.

$$\nabla \times H = J$$

$$J = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 3\, x \cos\beta + 6\, y \sin\alpha \end{vmatrix}$$

$$J = \frac{\partial}{\partial y}(3\, x \cos\beta + 6\, y \sin\alpha)\, \bar{a}_x - \frac{\partial}{\partial x}(3\, x \cos\beta + 6\, y \sin\alpha)\, \bar{a}_y$$

$$\mathbf{J = 6 \sin\alpha\, \bar{a}_x - 3 \cos\beta\, \bar{a}_y \quad A/m^2}$$ **Ans.** ☜ ☺☺

Example 4.16 *Given the conduction current density in a lossy dielectric as $J_C = 0.02 \sin 10^9\, t\ A/m^2$. Find the displacement current density if $\sigma = 10^3$ mho/m and $\varepsilon_r = 6.5$.*

☺ **Solution:**

$$J_C = 0.02 \sin 10^9\, t \quad \text{A/m}^2$$

$$\sigma = 10^3 \text{ mho/m}$$

$$\varepsilon_r = 6.5$$

$$J_C = \sigma E$$

$$E = \frac{J_C}{\sigma}$$

$$= \frac{0.02 \sin 10^9\, t}{10^3} = 2 \times 10^{-5} \sin 10^9\, t \quad \text{V/m}$$

$$\text{Displacement current density } J_D = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t}$$

$$\frac{\partial E}{\partial t} = 2 \times 10^{-5} \times 10^9 \cos 10^9\, t$$

$$J_D = \varepsilon \frac{\partial E}{\partial t} = \varepsilon_o \varepsilon_r \frac{\partial E}{\partial t}$$

$$= 8.854 \times 10^{-12} \times 6.5 \times 2 \times 10^{-5} \times 10^9 \cos 10^9 t \text{ A/m}$$

$$= \mathbf{115.1 \times 10^{-8} \cos 10^9 t \text{ A/m}}$$ **Ans.** ☜ ☺☺

Example 4.17 *Do the fields* $E = E_m \sin x \sin t\, \bar{a}_y$ *and* $H = \frac{E_m}{\mu_o} \cos x \cos t\, \bar{a}_z$ *satisfy Maxwell's equations?*

☺ ***Solution:*** $E = E_m \sin x \sin t\, \bar{a}_y, \quad H = \frac{E_m}{\mu_o} \cos x \cos t\, \bar{a}_z$

By Maxwell's equation,

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu_0 \frac{\partial H}{\partial t}$$

$$\nabla \times E = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_m \sin x \sin t & 0 \end{bmatrix}$$

$$= \bar{a}_x \left[-\frac{\partial}{\partial z} (E_m \sin x \sin t) \right] + \bar{a}_z \left[\frac{\partial}{\partial x} (E_m \sin x \sin t) \right]$$

$$= E_m \cos x \sin t\, \bar{a}_z$$

$$-\mu_o \frac{\partial H}{\partial t} = -\mu_o \frac{\partial}{\partial t} \left[\frac{E_m}{\mu_o} \cos x \cos t \right] \bar{a}_z$$

$$= -E_m \cos x (-\sin t)\, \bar{a}_z = E_m \cos x \sin t\, \bar{a}_z$$

Hence $\nabla \times E = -\mu \frac{\partial H}{\partial t}$

∴ E and H satisfy Maxwell's equations. ☺☺

Example 4.18 *State Faraday's laws. What are the different ways of emf generation? Explain with governing equation and suitable example for each.*

☺ ***Solution:*** Faraday's law states that the total electromagnetic force induced in a circuit is equal to the rate of decrease of the total magnetic flux linking the circuit.

$$v = -\frac{d\phi}{dt}$$

If N is the number of turns in the circuit, the total flux is N ϕ. Then

$$v = -N \frac{d\phi}{dt}$$

An emf can be generated in a closed conducting loop by three ways.

(*i*) A stationary loop in a time varying magnetic field.

(*ii*) A moving loop in a static magnetic field.

(*iii*) A moving loop in a time varying magnetic field.

(i) Stationary loop in a time varying magnetic field:

When the conducting loop is stationary in a time varying magnetic field 'B', the emf is induced in the loop. The induced emf is called transformer emf.

$$v = N \int_S \frac{\partial B}{\partial t}\, dS$$

where N is the number of turns in a loop.

If, $N = 1$

$$v = -\int_S \frac{\partial B}{\partial t}\, dS$$

It may be related to electric field E.

$$v = \oint E \cdot dl$$

Equating these two equations,

$$\oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS$$

This is the integral form of Faradays law.

By applying Stoke's theorem,

$$\oint E \cdot dl = \int_S (\nabla \times E) \cdot dS$$

$$\therefore \int_S (\nabla \times E)\, dS = -\int_S \frac{\partial B}{\partial t} \cdot dS$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This is the differential form of Faradays law. It states that a time varying magnetic field induces an electric field E whose curl is equal to the negative of the time derivation of B.

(ii) Moving loop in a static magnetic field:

Consider a conducting loop of length l moving across a static magnetic field B with a velocity v. The magnetic force F acting on any charged particle Q moving with a velocity v in a magnetic field B is given by

$$F = Q(v \times B)$$

The magnetic force is equivalent to the electrical force that would be exerted on the particle by an electric field E given by

$$E = \frac{F}{Q} = v \times B$$

The electric field E generated by the motion of the charged particle is called a motional electric field and it is perpendicular to v and B.

The induced voltage is induced emf and is given by

$$\int E \cdot dl = \int (v \times B)\, dl$$

Example: Electric generator and motor.

Permanent magnet is used to produce a static magnetic field B. In the case of electric generator, emf is induced when the loop rotates in the magnetic field. It converts mechanical energy into electrical energy. In the case of electric motor, when a current is passing through the loop, loop starts to rotate in magnetic field. It converts electrical energy into mechanical energy.

(iii) Moving loop in a time varying magnetic field:

When the conducting loop is moving in a time varying magnetic field B, the induced emf is the sum of transformer emf and motional emf ☺☺

$$\text{Total emf} = \oint E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_l (v \times B)\, dl$$

Example 4.19 *In free space, $H = 0.2 \cos(\omega t - \beta x)\, \vec{a_x}$ A/m. Find the total power passing through a circular disc of radius 5 cm.*

☺ ***Solution: Given:***

$$H = 0.2 \cos(\omega t - \beta x)\, \vec{a_x}$$

$$\eta = 120\pi$$

$$\text{Area} = \pi r^2 = \pi \times (0.05)^2$$

$$E = 120\pi \times 0.2 = 75.4 \text{ N/m}$$

$$\text{Average power } P_{av} = \frac{1}{2} E \cdot H = \frac{1}{2} \times 75.4 \times 0.2$$

$$P_{av} = 7.54 \text{ W/m}^2$$

$$\text{Total power } P = P_{av} \times \text{Area} = 7.54 \times \pi \times (0.05)^2$$

$$= \mathbf{0.059\ W\ or\ 59\ mW}$$ **Ans.** ☜ ☺☺

Example 4.20 *In a charge free non-magnetic dielectric region, the magnetic field is given by $\vec{H} = 5 \cos(10^9 t - 4y)\, \vec{a_z}$ A/m.*

Calculate the dielectric constant of the medium and also the displacement current density.

☺ ***Solution:*** From the equation $H = 5\cos(10^9 t - 4y)\, \bar{a}_z$

$$\omega = 2\pi f = 10^9$$

$$f = \frac{10^9}{2\pi}, \quad \beta = 4$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$v = f\lambda = \frac{10^9}{2\pi} \times \frac{\pi}{2} = \frac{10^9}{4} = 2.5 \times 10^8 \text{ m/s}$$

$$v = \frac{c}{\sqrt{\mu_r \varepsilon_r}}$$

$$\mu_r = 1$$

$$v = \frac{c}{\sqrt{\varepsilon_r}}$$

$$\varepsilon_r = \left(\frac{c}{v}\right)^2 = \left(\frac{3}{2.5}\right)^2$$

$$\varepsilon_r = 1.44$$

$$\nabla \times H = \frac{\partial D}{\partial t} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 5\cos(10^9 t - 4y) \end{vmatrix}$$

$$= \vec{a}_x \left[\frac{\partial}{\partial y} [(5\cos(10^9 t - 4y))] \right] - \vec{a}_y (0) + \vec{a}_z (0)$$

$$= 5\cos(10^9 t - 4y)(-4)$$

$$= \mathbf{-20\cos(10^9 t - 4y)\ A/m^2}$$ **Ans.** ☜ ☺☺

SUMMARY

Faraday's law $v = -N \frac{d\phi}{dt}$

Emf equation for transformer $E = 4.44 f N \phi_m$

Motional emf $e = B v l \sin\theta$

Induced emf in Faraday's disc $= \frac{\omega B a^2}{2}$

Force required to pull the magnet $F = \frac{B^2 A}{2\mu_0}$

Magnetic flux $= \frac{\text{mmf}}{\text{Reluctance}}$

Reluctance $\mathcal{R} = \frac{l}{\mu A}$ Henry^{-1}

Permeance $P = \frac{1}{\mathcal{R}} = \frac{\mu A}{l}$ Henry

Conduction current density $J_C = \sigma E$

Displacement current density $J_D = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t}$

Continuity equation $\nabla \cdot J = \frac{\partial \rho_v}{\partial t}$

Maxwell's Equations for conducting Medium

	Differential form	Integral form
I.	$\nabla \times H = J + \frac{\partial D}{\partial t}$ $\nabla \times H = \sigma E + \varepsilon \frac{\partial E}{\partial t}$	$\oint H \cdot dl = \iint_S \left(J + \frac{\partial D}{\partial t} \right) ds$ $\oint H \cdot dl = \iint_S \left(\sigma E + \varepsilon \frac{\partial E}{\partial t} \right) ds$
II.	$\nabla \times E = -\frac{\partial B}{\partial t}$ $\nabla \times E = -\mu \frac{\partial H}{\partial t}$	$\oint E \cdot dl = -\iint_S \frac{\partial B}{\partial t} \cdot ds$ $\oint E \cdot dl = -\mu \iint_S \frac{\partial H}{\partial t} \cdot ds$
III.	$\nabla \cdot D = \rho$	$\oiint_s D \cdot ds = \iiint_v \rho \, dv$
IV.	$\nabla \cdot B = 0$	$\oiint_s B \cdot ds = 0$

Maxwell's equations for free space

	Differential form	Integral form
I.	$\nabla \times H = \frac{\partial D}{\partial t} = \varepsilon \frac{\partial E}{\partial t}$	$\oint H \cdot dl = \iint_S \frac{\partial D}{\partial t} ds = \varepsilon \iint \frac{\partial E}{\partial t} \cdot ds$
II.	$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$	$\oint E \cdot dl = -\iint_S \frac{\partial B}{\partial t} \cdot ds$ $\oint E \cdot dl = -\mu \iint_S \frac{\partial H}{\partial t} \cdot ds$
III.	$\nabla \cdot D = 0$	$\oiint_s D \cdot ds = 0$
IV.	$\nabla \cdot B = 0$	$\oiint_s B \cdot ds = 0$

Maxwell's Equation in Phasor form

	Differential form	Integral form
I.	$\nabla \times H = (\sigma + j\omega\varepsilon) E$	$\oint H \cdot dl = \iint (\sigma + j\omega\varepsilon) E \cdot ds$
II	$\nabla \times E = - j\omega\mu H$	$\oint E \cdot dl = -\mu \iint j\omega H \cdot ds$
III.	$\nabla \cdot D = \rho$	$\oint D \cdot ds = \iiint \rho \, dv$
IV.	$\nabla \cdot B = 0$	$\oint B \cdot ds = 0$

TWO MARK QUESTIONS AND ANSWERS

1. ***Define current density.***

Current density is defined as current per unit area. It is denoted by

$$J = \frac{I}{A} \text{ Amp/m}^2.$$

2. ***What is meant by displacement current?***

Displacement current is nothing but the current flows through the capacitor.

Displacement current density is given by

$$J_D = \frac{\partial D}{\partial t} \text{ A/m}^2$$

3. ***Given for copper $\sigma = 5.8 \times 10^7$ mho/m, and E = 1 V/m, find for a circular conductors of A = 1 cm² cross sections area (a) J, (b) I.***

(*a*)
$$J = \sigma E$$
$$= 5.8 \times 10^7 \times 1 = 5.8 \times 10^7 \text{ Amp/m}^2$$

(*b*)
$$I = J A$$
$$= 5.8 \times 10^7 \times 1 \times (10^{-2})^2 = 5.8 \times 10^3 \text{ Amps}$$

4. ***Write the point form of continuity equation and explain its significance.***

$$\nabla \cdot J = - \frac{\partial \rho_v}{\partial t}$$

Since the charge is conserved, the outward flux of J must therefore be equal to the rate of loss of charge within the volume.

5. ***State point form of Ohm's law.***

Point form of ohm's law states that the field strength within a conductor is proportional to the current density.

$$J \propto E$$
$$J = \sigma E$$

where, σ = Conductivity of the material

6. ***State Faraday's law of electromagnetic induction.***

Faraday's law states that electromagnetic force induced in a circuit is equal to the rate of change of magnetic flux linking the circuit.

$$\text{emf} = \frac{d\phi}{dt}$$

7. ***State Lenz's law.***

Lenz's law states that the induced emf in a circuit produces a current, which oppose the change in magnetic flux producing it

$$\text{emf} = - \frac{d\phi}{dt}$$

8. ***In a solenoid with an inductance of 5 mH current is increasing at the rate of 100 A/sec. What is the value of induced emf?***

$$\text{emf} = L \frac{di}{dt} = 5 \times 10^{-3} \times 100 = 0.5 \text{ V}$$

9. ***A conductor of 1 m length is dragged with a velocity of 100 m/sec, perpendicular to a field of 1 T. What is the value of emf induced?***

l = 1 m, v = 100 m/s, B = 1 Wb/m² (T), $\theta = \pi/2$

$$e = B l v \sin\theta$$
$$= 1 \times 1 \times 100 = 100 \text{ volts}$$

10. ***Distinguish between transformer emf and motional emf.***

The emf induced in a stationary conductor due to the change in flux linked with it, is called transformer emf or static induced emf.

$$emf = - \iint \frac{\partial B}{\partial t} \cdot ds \qquad \textit{e.g.}, \text{ transformer}$$

The emf induced due to the movement of conductor in a magnetic field is called motional emf or dynamic induced emf.

$$emf = \oint_c v \times B \cdot dl \qquad e.g., \text{ generator}$$

11. ***What is the equation for total emf due to motion or transformer action?***

The equation for total emf due to motion or transformer action is

$$E = \int (v \times B) \cdot dh - \int_s \frac{\partial B}{\partial t} \cdot ds$$

12. ***Write the Maxwell's equation from Ampere's law both in integral and point forms.***

Maxwell's equations from Ampere's law,

$$\oint H \cdot dl = \iint_S \left(J + \frac{\partial D}{\partial t} \right) ds \qquad \text{(Integral form)}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \qquad \text{(Point form)}$$

13. ***Write the Maxwell's equation from Faraday's law both in integral and point forms.***

Maxwell's equation from Faraday's law,

$$\oint E \cdot dl = -\iint \frac{\partial B}{\partial t} \cdot ds \qquad \text{(Integral form)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{(Point form)}$$

14. ***Write the Maxwell's equations from Gauss's law in integral form. Maxwell's equations from Gauss's law in integral form.***

$$\oiint_s D \cdot ds = \iiint_v \rho \, dv \qquad \text{(Electric field)}$$

$$\oiint_s B \cdot ds = 0 \qquad \text{(Magnetic field)}$$

15. ***Write down the Maxwell's equation from Gauss's law in point form. Maxwell's equations from Gauss's law in point form.***

$$\nabla \cdot D = \rho \qquad \text{(Electric field)}$$

$$\nabla \cdot B = 0 \qquad \text{(Magnetic field)}$$

16. ***Write down the Maxwell's equation from electric Gauss's law in integral and point form.***

Maxwell's equation from electric Gauss's law,

$$\oiint D \cdot ds = \iiint_v \rho \, dv \qquad \text{(Integral form)}$$

$$\nabla \cdot D = \rho \qquad \text{(Point form)}$$

17. ***Write down the Maxwell's equations from magnetic Gauss's law in integral and point form.***

Maxwell's equation from magnetic Gauss's law

$$\oiint_s B \cdot ds = 0 \qquad \text{(Integral form)}$$

$$\nabla \cdot B = 0 \qquad \text{(Point form)}$$

18. ***Write the Maxwell's equations in integral form.***

Maxwell's equations in integral form

$$\oint H \cdot dl = \iint_S \left(J + \frac{\partial D}{\partial t} \right) ds$$

$$\oint E \cdot dl = -\iint_S \frac{\partial B}{\partial t} \, ds$$

$$\oiint D \cdot ds = \iiint_v \rho \, dv$$

$$\oiint B \cdot ds = 0$$

19. ***Write down the Maxwell's equations in point form.***

Maxwell's equations in point form

$$\nabla \times H = J + \frac{\partial D}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

20. ***Write down the Maxwell's equations in point phasor forms.***

$$\nabla \times H = J + j\omega D = (\sigma + j\omega\varepsilon)E$$

$$\nabla \times E = -j\omega B = -j\omega\mu H$$

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

21. ***Write down the Maxwell's equations in integral phasor form.***

$$\oint H \cdot dl = \iint_S (J + j\omega D)\, ds = \iint_S (\sigma + j\omega\varepsilon) E\, ds$$

$$\oint E \cdot dl = -\iint_S j\omega B \cdot ds = -\mu \iint j\omega H\, ds$$

$$\oiint_s D \cdot ds = \iiint \rho\, dv$$

$$\oiint B \cdot ds = 0$$

22. ***Write down the Maxwell's equations for free space in integral form.***

$$\oint H \cdot dl = \iint_S \frac{\partial D}{\partial t} \cdot ds$$

$$\oint E \cdot dl = -\iint_S \frac{\partial B}{\partial t} \cdot ds$$

$$\oiint_s D \cdot ds = 0$$

$$\oiint_s B \cdot ds = 0$$

23. ***Write down the Maxwell's equations for free space in point form.***

$$\nabla \times H = \frac{\partial D}{\partial t}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot D = 0$$

$$\nabla \cdot B = 0$$

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24. ***Explain why $\nabla \cdot B = 0$.***

$\nabla \cdot B = 0$ states that there is no magnetic charges. The net magnetic flux emerging through any closed surface is zero.

25. ***Explain why $\nabla \times E = 0$.***

In a region in which there is no time changing magnetic flux, the voltage around the loop would be zero.

By Maxwell's equation

$$\nabla \times E = -\frac{\partial B}{\partial t} = 0 \quad \text{(irrotational)}$$

26. ***Explain why $\nabla \cdot D = 0$.***

In a free-space, there is no charge enclosed by the medium. The volume charge density is zero.

By Maxwell's equation, $\nabla \cdot D = \rho = 0$

EXERCISE

1. State Faraday's law. What are the different ways of emf generation? Explain it with governing equation.
2. Derive the emf equations for transformer and motional emf.
3. Discuss the transformer emf and motional emf.
4. Describe the Faraday's disc generator and derive the equation for its induced emf.
5. Derive the expression for force required to pull the magnet.
6. State Maxwell's equations and obtain them in differential form. Also derive them for harmonically varying field.
7. State Maxwell's equations and obtain them in integral and differential form.
8. Derive and explain the Maxwell's equations in point and integral form using Ampere's circuital law and Faraday's law.
9. Derive the Maxwell's equations for free space in integral and point forms and explain them.

10. Derive Maxwell's equations from Faraday's law and Gauss's law and explain them.
11. Derive the Maxwell's equations in phasor differential form.
12. Derive the Maxwell's equation in phasor integral form.
13. Derive the Maxwell's equations in phasor form for free space.
14. State and derive the Maxwell's equations for free space in integral and point forms for time varying field.
15. Obtain the Maxwell's equations for conducting medium and free space in integral and point forms.
16. Obtain the Maxwell's equations for conducting medium and free space in integral and point forms for time varying field.
17. State and prove boundary conditions by the application of Maxwell's equations.
18. Find the amplitude of the displacement current density inside a capacitor where $\varepsilon_r = 600$ and $D = 3 \times 10^{-6} \sin(6 \times 10^6 t - 0.346 x)\, \bar{u}_z$ c/m².

□□

5

ELECTROMAGNETIC WAVES

5.1. ELECTROMAGNETIC WAVE EQUATION

5.1.1. Wave Equation for Conducting Medium

The application of Maxwell's equations is the prediction of existence of electromagnetic wave. Electromagnetic wave equation can be obtained from Maxwell's equations.

The Maxwell's equation from Faraday's law in point form is given by

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t}$$

Take curl on both sides,

$$\nabla \times \nabla \times E = -\mu \nabla \times \frac{\partial H}{\partial t} \quad \ldots (1)$$

But Maxwell's equation from Ampere's law in point form is

$$\nabla \times H = J + \frac{\partial D}{\partial t} = \sigma E + \varepsilon \frac{\partial E}{\partial t}$$

Differentiating,

$$\nabla \times \frac{\partial H}{\partial t} = \frac{\partial \nabla \times H}{\partial t} = \frac{\partial}{\partial t}\left(\sigma E + \varepsilon \frac{\partial E}{\partial t}\right)$$

$$\nabla \times \frac{\partial H}{\partial t} = \sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2} \quad \ldots (2)$$

Substituting the equation (2) in equation (1),

$$\nabla \times \nabla \times E = -\mu \left[\sigma \frac{\partial E}{\partial t} + \varepsilon \frac{\partial^2 E}{\partial t^2}\right]$$

$$= -\mu\sigma\frac{\partial E}{\partial t} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} \quad \dots (3)$$

But according to the identity,

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E \quad \dots (4)$$

$$\text{But} \quad \nabla \cdot E = \frac{1}{\varepsilon}\nabla \cdot D$$

Since there is no net charge within the conductor, the charge density $\rho = 0$.

$$\nabla \cdot D = 0$$

$$\nabla \cdot E = 0$$

Then equation (4) becomes,

$$\nabla \times \nabla \times E = -\nabla^2 E \quad \dots (5)$$

Comparing the equations (3) and (5)

$$\nabla^2 E = \mu\sigma\frac{\partial E}{\partial t} + \mu\varepsilon\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E - \mu\sigma\frac{\partial E}{\partial t} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} = 0 \quad \dots (6)$$

This is the wave equation in terms of electric field E.

The wave equation in terms of magnetic field H is obtained in a similar manner as follows.

The Maxwell's equation from Ampere's law in point form is given by

$$\nabla \times H = \sigma E + \varepsilon\frac{\partial E}{\partial t}$$

Taking curl on both sides,

$$\nabla \times \nabla \times H = \sigma\nabla \times E + \varepsilon\nabla \times \frac{\partial E}{\partial t} \quad \dots (7)$$

But Maxwell's equation from Faraday's law

$$\nabla \times E = -\mu\frac{\partial H}{\partial t}$$

Differentiating with respect to 't',

$$\nabla \times \frac{\partial E}{\partial t} = -\mu\frac{\partial^2 H}{\partial t^2}$$

Substituting the values of $\nabla \times E$ and $\nabla \times \frac{\partial E}{\partial t}$ in equation (7),

$$\nabla \times \nabla \times H = -\mu\sigma\frac{\partial H}{\partial t} - \mu\varepsilon\frac{\partial^2 H}{\partial t^2} \quad \dots (8)$$

But the identity is $\quad \nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H$

$$\text{But} \quad \nabla \cdot B = \mu\nabla \cdot H = 0$$

$$\text{Then,} \quad \nabla \times \nabla \times H = -\nabla^2 H \quad \dots (9)$$

Comparing equations (8) and (9)

$$\nabla^2 H = \mu\sigma\frac{\partial H}{\partial t} + \mu\varepsilon\frac{\partial^2 H}{\partial t^2}$$

$$\nabla^2 H - \mu\sigma\frac{\partial H}{\partial t} - \mu\varepsilon\frac{\partial^2 H}{\partial t^2} = 0 \quad \dots (10)$$

This is the wave equation in terms of magnetic field H.

5.1.2. Wave Equation for Free Space

For free space (dielectric medium) the conductivity of the medium is zero. (*i.e.,* $\sigma = 0$) and there is no charge containing in it (*i.e.,* $\rho = 0$). The electromagnetic wave equations for free space can be obtained from Maxwell's equations.

The Maxwell's equation from Faraday's law for free space in point form is

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t}$$

Taking curl on both sides,

$$\nabla \times \nabla \times E = -\mu\nabla \times \frac{\partial H}{\partial t} \quad \dots (11)$$

But Maxwell's equation from Ampere's law for free space in point form is

$$\nabla \times H = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t}$$

$$\text{Then,} \quad \nabla \times \frac{\partial H}{\partial t} = \frac{\partial \nabla \times H}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon\frac{\partial E}{\partial t}\right)$$

$$\nabla \times \frac{\partial H}{\partial t} = \varepsilon\frac{\partial^2 E}{\partial t^2} \quad \dots (12)$$

Substituting the equation (12) in equation (11),

$$\nabla \times \nabla \times E = -\mu\varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \ldots (13)$$

But the identity is given by,

$$\nabla \times \nabla \times E = \nabla(\nabla \cdot E) - \nabla^2 E$$

$$\text{But} \quad \nabla \cdot E = \frac{1}{\varepsilon}\nabla \cdot D = \frac{\rho}{\varepsilon} = 0$$

$$\text{Then,} \quad \nabla \times \nabla \times E = -\nabla^2 E \qquad \ldots (14)$$

Comparing the equations (13) and (14),

$$\nabla^2 E = \mu\varepsilon \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E - \mu\varepsilon \frac{\partial^2 E}{\partial t^2} = 0 \qquad \ldots (15)$$

This is the wave equation for free space in terms of electric field.

The wave equation for free space in terms of magnetic field H is obtained in a similar manner as follows.

The Maxwell's equation from Ampere's law for free space in point form is given by

$$\nabla \times H = \varepsilon \frac{\partial E}{\partial t}$$

Taking curl on both sides

$$\nabla \times \nabla \times H = \varepsilon \nabla \times \frac{\partial E}{\partial t} \qquad \ldots (16)$$

But Maxwell's equation from Faraday's law

$$\nabla \times E = -\mu \frac{\partial H}{\partial t}$$

$$\text{Differentiating,} \quad \nabla \times \frac{\partial E}{\partial t} = -\mu \frac{\partial^2 H}{\partial t^2} \qquad \ldots (17)$$

Substituting equation (17) in equation (16)

$$\nabla \times \nabla \times H = -\mu\varepsilon \frac{\partial^2 H}{\partial t^2} \qquad \ldots (18)$$

But the identity is given by

$$\nabla \times \nabla \times H = \nabla(\nabla \cdot H) - \nabla^2 H \qquad \ldots (19)$$

$$\text{But} \quad \nabla \cdot H = \frac{1}{\mu}\nabla \cdot B = 0$$

$$\text{Then,} \quad \nabla \times \nabla \times H = -\nabla^2 H \qquad \ldots (20)$$

Comparing equations (18) and (20)

$$\nabla^2 H = \mu\varepsilon \frac{\partial^2 H}{\partial t^2}$$

$$\nabla^2 H - \mu\varepsilon \frac{\partial^2 H}{\partial t^2} = 0 \qquad \ldots (21)$$

This is the wave equation for free space in terms of H.

For free space $\mu_r = 1$ and $\varepsilon_r = 1$ (air), then wave equation becomes

$$\nabla^2 H - \mu_0 \varepsilon_0 \frac{\partial^2 H}{\partial t^2} = 0$$

$$\mu_0 \varepsilon_0 = 4\pi \times 10^{-7} \times \frac{1}{36\pi \times 10^{-9}} = \frac{1}{9 \times 10^{16}}$$

$$\frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s} = v_0$$

where v_0 is the velocity of light.

Then the wave equation,

$$\nabla^2 H - \frac{1}{v_0^2}\frac{\partial^2 H}{\partial t^2} = 0 \quad \text{or} \quad \nabla^2 E - \frac{1}{v_0^2}\frac{\partial^2 E}{\partial t^2} = 0$$

5.2. UNIFORM PLANE WAVE

If the phase of a wave is the same for all points on a plane surface, it is called plane wave. If the amplitude is also constant in a plane wave, it is called uniform plane wave. The properties of uniform plane waves are given as follows:

1. At every point in space electric field (E) and magnetic field (H) are perpendicular to each other and to the direction of travel.
2. The fields vary harmonically with time and at the same frequency, everywhere in space.

3. Each field has the same direction, magnitudes and phase at every point in any plane perpendicular to the direction of wave travel.

If the electric field is in x direction and the magnetic field is in y direction, then the wave is travelling in z direction.

The wave equation for free space is given by

$$\nabla^2 E = \mu \varepsilon \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E}{\partial t^2}$$

Consider electric field E varies in x direction and E is independent of directions y and z, then wave equation becomes

$$\frac{\partial^2 E}{\partial x^2} = \mu \varepsilon \frac{\partial^2 E}{\partial t^2} \qquad \left[\because \frac{\partial^2 E}{\partial y^2} = \frac{\partial^2 E}{\partial z^2} = 0\right]$$

It can be written as in terms of the components of E.

$$\frac{\partial^2 E_x}{\partial x^2} = \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu \varepsilon \frac{\partial^2 E_y}{\partial t^2}$$

$$\frac{\partial^2 E_z}{\partial x^2} = \mu \varepsilon \frac{\partial^2 E_z}{\partial t^2}$$

For free space there is no charge density.

$$\nabla \cdot D = \varepsilon \nabla \cdot E = 0$$

$$\nabla \cdot E = 0$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0$$

For uniform plane wave, E is independent of y and z.

$$\text{Then,} \quad \frac{\partial E_x}{\partial x} = 0$$

This equation shows that there is no variation of E_x in the x direction.

Differentiating this equation with respect to 't',

$$\frac{\partial^2 E}{\partial x^2} = 0$$

It requires that either E_x be zero or constant.

If E_x is constant, it would not part of wave motion and so E_x must be zero. Therefore a uniform plane wave propagating (travelling) in the x direction has no x component of E. A similar analysis would show that there is no x component of H.

$$\nabla \cdot B = \mu \nabla \cdot H = 0$$

$$\nabla \cdot H = 0$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

Since H is propagating in x direction, it is independent of y and z. Then

$$\frac{\partial H_x}{\partial x} = 0 \text{ and } \frac{\partial^2 H_x}{\partial x^2} = 0$$

Since H_x is not a constant, H_x must be zero.

$\therefore$ $H_x = 0$ for uniform plane wave.

5.2.3. Characteristic Impedance

Consider the plane wave propagating in x direction. The wave equation for free space is

$$\frac{\partial^2 E}{\partial x^2} = \mu \varepsilon \frac{\partial^2 E}{\partial t^2}$$

The general solution of this differential equation is in the form

$$E = f_1(x - v_0 t) + f_2(x + v_0 t)$$

$$\text{where} \quad v_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

f_1 and f_2 are any functions of $(x - v_0 t)$ and $(x + v_0 t)$ respectively.

The solution of wave equation consists of two waves, one travelling in positive direction and other travelling in negative direction. Consider the wave travel in positive direction alone.

$$f_2(x + v_0 t) = 0 \qquad \text{(negative direction)}$$

The general solution of wave equation becomes

$$E = f(x - v_0 t)$$

$$\nabla \times E = \begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \vec{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \vec{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \vec{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

Since wave travelling in x direction, E and H are both independent of y and z.

i.e., $E_x = H_x = 0$ and $\frac{\partial E}{\partial y} = \frac{\partial E}{\partial z} = 0$

$$\nabla \times E = -\frac{\partial E_z}{\partial x}\vec{y} + \frac{\partial E_y}{\partial x}\vec{z}$$

Similarly, $\nabla \times H = -\frac{\partial H_z}{\partial x}\vec{y} + \frac{\partial H_y}{\partial x}\vec{z}$

But, $\nabla \times H = \varepsilon \frac{\partial E}{\partial t}$

Comparing these two equations,

$$-\frac{\partial H_z}{\partial x}\vec{y} + \frac{\partial H_y}{\partial x}\vec{z} = \varepsilon\left[\frac{\partial E_y}{\partial t}\vec{y} + \frac{\partial E_z}{\partial t}\vec{z}\right] \quad [\because E_x = 0]$$

Equating $\vec{y}$ and $\vec{z}$ terms

$$-\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t}$$

$$\frac{\partial H_y}{\partial x} = \varepsilon \frac{\partial E_z}{\partial t}$$

From second Maxwell's equation for free space

$$\nabla \times E = -\mu \frac{\partial E}{\partial t}$$

But $\nabla \times E = -\frac{\partial E_z}{\partial x}\vec{y} + \frac{\partial E_y}{\partial x}\vec{z}$

Equating these two equations,

$$-\frac{\partial E_z}{\partial x}\vec{y} + \frac{\partial E_y}{\partial x}\vec{z} = -\mu\left[\frac{\partial H_y}{\partial t}\vec{y} + \frac{\partial H_z}{\partial t}\vec{z}\right] \quad [\because H_x = 0]$$

Equating $\vec{y}$ and $\vec{z}$ terms

$$\frac{\partial E_z}{\partial x} = \mu \frac{\partial H_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t}$$

Let the solution of this equation is given by

$$E_y = f(x - v_0 t)$$

Differentiating $\frac{\partial E_y}{\partial t} = \frac{\partial f}{\partial(x - v_0 t)} \cdot \frac{\partial(x - v_0 t)}{\partial t}$

$$= f'(x - v_0 t)(-v_0)$$

Simplify $f'(x - v_0 t)$ can be written as f'.

$$\frac{\partial E_y}{\partial t} = -v_0 f'$$

But, $-\frac{\partial H_z}{\partial x} = \varepsilon \frac{\partial E_y}{\partial t}$

$$\frac{\partial H_z}{\partial x} = -\varepsilon(-v_0 f') = v_0 \varepsilon f'$$

$$= \frac{1}{\sqrt{\mu \varepsilon}} \varepsilon f'$$

$$\frac{\partial H_z}{\partial x} = \sqrt{\frac{\varepsilon}{\mu}} f'$$

Integrating $H_z = \sqrt{\frac{\varepsilon}{\mu}} \int f' \, dx$

$$H_z = \sqrt{\frac{\varepsilon}{\mu}} f = \sqrt{\frac{\varepsilon}{\mu}} E_y$$

$$\frac{E_y}{H_z} = \sqrt{\frac{\mu}{\varepsilon}}$$